MTH161 Workshop 3: limits at infinity, continuity, intro to derivatives

Discussion Questions: Discuss the following question(s) with your group.

- What does it mean to say that $f(x)$ is continuous at $x = a$ (be precise)?
- What does it mean to say that a graph has a horizontal asymptote? Can the graph of a function intersect a horizontal asymptote?

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1. Consider the functions

$$f(x) = \begin{cases} \dfrac{x^2+x}{x^3+x}, & x \neq 0 \\ a, & x = 0 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ a, & x = 0. \end{cases}$$

(a) Does there exist a real number a for which the function f is continuous everywhere? Justify your answer.

(b) Same question, but for g.

2. (a) Find functions $f(t)$ and $g(t)$ such that $\lim_{t\to\infty} f(t) = \infty = \lim_{t\to\infty} g(t)$, and $\lim_{t\to\infty} \dfrac{f(t)}{g(t)} = 10$.

(b) Find functions $f(t)$ and $g(t)$ such that $\lim_{t\to\infty} f(t) = \infty = \lim_{t\to\infty} g(t)$, and $\lim_{t\to\infty} \dfrac{f(t)}{g(t)} = \infty$.

(c) Find functions $f(t)$ and $g(t)$ such that $\lim_{t\to\infty} f(t) = \infty = \lim_{t\to\infty} g(t)$, and $\lim_{t\to\infty} \dfrac{f(t)}{g(t)} = 0$.

3. (a) Explain, in your own words, what the Intermediate Value Theorem says.

(b) Use the Intermediate Value Theorem to show the following equation has a solution:

$$\tan^{-1}(x) + x - 1 = 0.$$

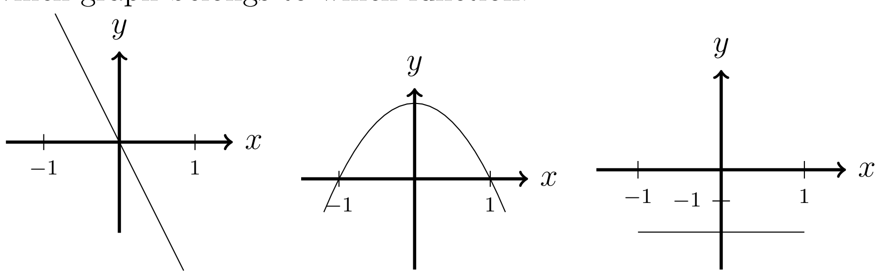
4. The *binomial theorem* tells us that for a positive integer n, that $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are called *binomial coefficients.* As it turns out, the binomial coefficients for n are given by the n^{th} row of *Pascal's triangle* (if you've never heard of that, Google it! or ask your group or TA). For example, the 3^{rd} row of Pascal's triangle is 1, 3, 3, 1 and these are the coefficients on the terms of $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

(a) Expand $(x+y)^5$.

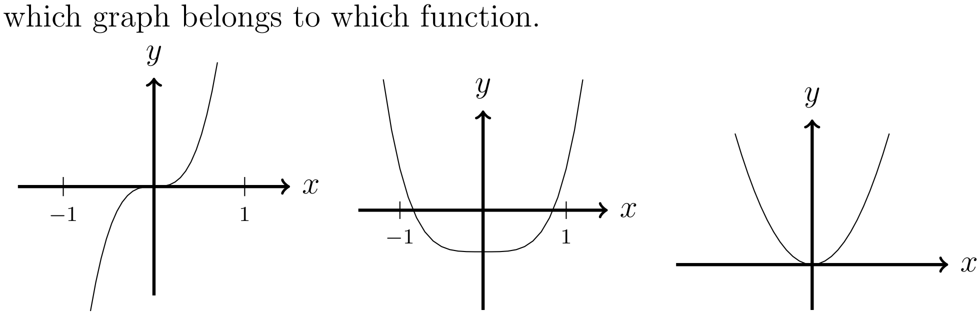
(b) Let $f(x) = x^5$ and use the definition of derivative $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ to show $f'(x) = 5x^4$. Hint: part (a) should help.

5. The **vertex** of a parabola is the point at which the tangent line is horizontal. Show that the vertex of the graph of $y = ax^2 + bx + c$ has x-coordinate $-\dfrac{b}{2a}$.

6. (a) Below are the graphs of $f(x)$, $f'(x)$, and $f''(x)$ for some function $f(x)$. Determine which graph belongs to which function.



(b) Below are the graphs of $g(x)$, $g'(x)$, and $g''(x)$ for some function $g(x)$. Determine which graph belongs to which function.



(c) The graph below shows $y = h(x)$ for some function h. Sketch the graph of $h'(x)$.