MTH 161

Midterm 2

March 24, 2022

Name: Solutions

UR ID: ____________________

Circle your instructor's Name:

Joshua Sumpter   Tritium Shen

Instructions

- The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.
- For each problem, please circle or box your final answer. We will judge your work outside the box as well (unless specified otherwise) so you still need to show your work and justify your answers. **You may not receive full credit for a correct answer if insufficient work is shown, insufficient justification is given, or we are unable to read what you have written.**
- In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$. However, known values of functions should be evaluated, for example, $\ln(e), \sin(\pi), e^0$.
- This exam is out of 100 points. You are responsible for checking that this exam has all 10 pages, including this one.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

____

____

____

YOUR SIGNATURE: ____________________

Some Volume Formulas

- Sphere: $V = \frac{4\pi}{3}r^3$.
- Circular Cone: $V = \pi r^2 \frac{h}{3}$.
- Cylinder: $V = \pi r^2 h$.
- Cube: $V = s^3$.
- Rectangular Prism (box): $V = lwh$.

Trig Identities

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $1 + \tan^2(\theta) = \sec^2(\theta)$
- $1 + \cot^2(\theta) = \csc^2(\theta)$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$
- $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$
- $\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
- $\sin(a)\sin(b) = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$
- $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
- $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

1. (18 points) Let $f(x) = \dfrac{x+4}{5-x}$.

(a) (10 points) Use the limit definition of the derivative to find $f'(x)$.

$$f(x) = \frac{x+4}{5-x} = \frac{x-5+9}{5-x} = -1 + \frac{9}{5-x}$$

$$\frac{f(x+h)-f(x)}{h} = \left(\frac{1}{h}\right)\left[\frac{9}{5-(x+h)} - \frac{9}{5-x}\right] = \frac{1}{h}\left[\frac{9(5-x) - 9(5-(x+h))}{(5-x)(5-(x+h))}\right]$$

$$= \frac{1}{h}\left[\frac{(45-9x)-(45-9x-9h)}{(5-x)(5-(x+h))}\right]$$

$$= \frac{1}{h}\left(\frac{9h}{(5-x)(5-(x+h))}\right)$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{9h}{(5-x)(5-(x+h))}\right)$$

$$= \lim_{h\to 0} \frac{9}{(5-x)(5-(x+h))}$$

$$= \frac{9}{(5-x)^2}$$

(b) (5 points) Find $f'(x)$ using derivative rule (i.e. the quotient rule) and verify that your answer from part (a) is correct.

$g(x) = x+4 \quad g'(x) = 1 \qquad f(x) = \dfrac{g(x)}{h(x)}$

$h(x) = 5-x \quad h'(x) = -1$

$$f'(x) = \frac{h(x)g'(x) - h'(x)g(x)}{[h(x)]^2} = \frac{(5-x)(1) - (-1)(x+4)}{(5-x)^2}$$

$$= \frac{9}{(5-x)^2}$$

(c) (3 points) Find the equation of the line tangent to the graph $y = f(x)$ at $(1, 2)$.

Note $f(1) = 5/4 \neq 2$, $f'(1) = 9/16$

Accepted answers:

1) As written $y = 9/16(x-1) + 2$

2) As it should be: $y = 9/16(x-1) + 5/4$

3) f does not have

Technically f does not have a tangent line at (1,2 since (1,2) is not on the graph of

2. (30 points) Compute the following derivatives.

(a) $\dfrac{d}{dx} \dfrac{x^4 + \sqrt{x} + \ln(x^x)}{x}$ $= \dfrac{d}{dx}\left(x^3 + x^{-1/2} + \ln(x)\right)$

$= 3x^2 - \frac{1}{2}x^{-3/2} + \frac{1}{x}$

(b) $\dfrac{d}{dx}(3x^2 + 4)e^x$

$= 6x\, e^x + (3x^2+4)e^x$

(c) $\dfrac{d}{dx} \dfrac{x\sin(x)}{(3-x)^2}$

$f(x) = x\sin(x) \quad f'(x) = \sin(x) + x\cos(x)$

$g(x) = (3-x)^2 \quad g'(x) = -2(3-x)$

$= \dfrac{d}{dx} \dfrac{f(x)}{g(x)} = \dfrac{gf' - g'f}{g^2}$

$= \dfrac{(3-x)^2(\sin(x) + x\cos(x)) + 2(3-x)x\sin(x)}{(3-x)^4}$

(d) $\frac{d}{dx}\ \ln(e^{3x} + \cos^2(x) + 2)$

$$= \frac{\frac{d}{dx}\left(e^{3x} + \cos^2(x) + 2\right)}{e^{3x} + \cos^2(x) + 2} = \frac{3e^{3x} - 2\sin(x)\cos(x)}{e^{3x} + \cos^2(x) + 2}$$

(e) $\frac{d}{dx}\ \arctan(e^x + 4x^2)$ (Possible Hint: $\arctan(x) = \tan^{-1}(x)$)

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\, e^x + 4x^2 = e^x + 8x$$

By chain Rule:

$$\frac{d}{dx}\arctan(e^x + 4x^2) = \frac{e^x + 8x}{1 + (e^x + 4x^2)^2}$$

(f) $\frac{d}{dx}\ \sin(\cos^3(x^4))$

Let $U = x^4$, $V = \cos(U)$, $W = V^3$, $f = \sin(W)$.

$f = \sin(\cos^3(x^4))$

$$\frac{df}{dx} = \frac{df}{dW} \cdot \frac{dW}{dV} \cdot \frac{dV}{dU} \cdot \frac{dU}{dx} = \cos(W)\, 3V^2 (-\sin(U))\, 4x^3$$

$$= -\cos(\cos^3(x^4))\, 3\cos^2(x^4)\sin(x^4)\, 4x^3$$

$$= -12x^3\cos(\cos^3(x^4))\cos^2(x^4)\sin(x^4)$$

3. (10 points) Find the derivative of the following function:

$$f(x) = \frac{2^x \cdot \sin(x) \cdot x^{\ln(x)}}{\sqrt{x} \cdot e^{\cos(x)}}$$

Hint: Use logarithmic differentiation.

$$\ln(f(x)) = \ln\left(\frac{2^x \sin(x)\, x^{\ln(x)}}{\sqrt{x} \cdot e^{\cos(x)}}\right)$$

$$= x\ln(2) + \ln(\sin(x)) + \ln(x)\ln(x) - \frac{1}{2}\ln(x) - \cos(x)$$

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)} = \ln(2) + \frac{\cos(x)}{\sin(x)} + \frac{2\ln(x)}{x} - \frac{1}{2x} + \sin(x)$$

$$f'(x) = \frac{2^x \sin(x)\, x^{\ln(x)}}{\sqrt{x} \cdot e^{\cos(x)}}\left[\ln(2) + \frac{\cos(x)}{\sin(x)} + \frac{2\ln(x)}{x} - \frac{1}{2x} + \sin(x)\right]$$

4. (16 points) A particle travels along a line with respect to time (in seconds) according to the following position function:

$$s(t) = 3t^4 - 16t^3 + 18t^2 \quad (t \geq 0)$$

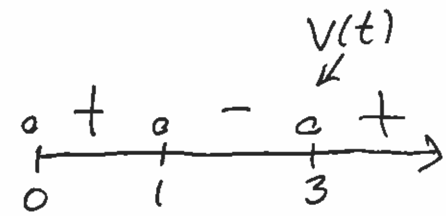
where $s(t)$ is in terms of meters.

(a) (6pts) When is the particle moving in the positive direction? When is the particle moving in the negative direction?

$V(t) = 12t^3 - 48t^2 + 36t$

$= 12t(t^2 - 4t + 3)$

$= 12t(t-1)(t-3)$



Positive direction: $(0, 1) \cup (3, \infty)$

Negative direction: $(1, 3)$

(b) (5pts) Find the total distance traveled by the particle during the first 2 seconds.

total distance = (distance traveled in pos. dir.) + (distance traveled in neg. dir.)

$= (s(1) - s(0)) + (s(1) - s(2))$

$= 2s(1) - s(0) - s(2) = 2(5) - 0 - (48 - 128 + 72)$

$= 2(5) + 8 = 18 \text{ m}$

(c) (5pts) Find the acceleration of the particle at $t = 1$ sec.

$a(t) = V'(t) = 36t^2 - 96t + 36$

$a(1) = 36 - 96 + 36$

$= -24 \text{ m/sec}^2$

5. (16 points) Consider the curve defined by $y^2 + xy = x^3 + x$

(a) (6pts) Find $\frac{dy}{dx}$ by using implicit differentiation.

$$2yy' + y + xy' = 3x^2 + 1$$

$$y' = \frac{3x^2+1-y}{2y+x}$$

(b) (6pts) Find $\frac{d^2y}{dx^2}$ (also known as y''). You may leave your answer in terms of x, y, and y'.

$$y'' = \frac{d}{dx} y' = \frac{d}{dx}\left(\frac{3x^2+1-y}{2y+x}\right)$$

$$= \frac{(2y+x)(6x-y') - (2y'+1)(3x^2+1-y)}{(2y+x)^2}$$

Part (c) is on the next page →

(c) (4pts) Find the equations of all the lines tangent to the curve when $x = 0$. Hint: There are two such tangent lines.

This Hint is wrong! (I meant to have $x=1$)

When $x=0$, $y^2=0$, so $y=0$.

- This means there is only on point corresponding to $x=0$, namely $(0,0)$.
- The corresponding tangent line is the vertical tangent line $x=0$.

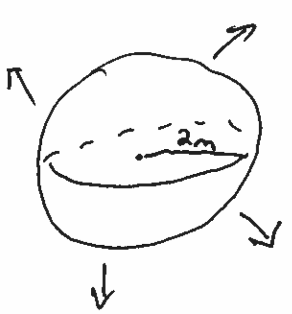
To make up for this error full points will be given to any student who attemps the problem.

6. (10 points) A large, spherical balloon is being filled with air at a rate of 3 m^3/sec. How fast is the surface area of the balloon changing when the radius is equal to 2 meters? Recall that the surface area of a sphere with radius r is equal to $4\pi r^2$.

Let V be the Volume of the balloon, r be the radius, & S be the surface area.

Then $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$.

- We know $\frac{dV}{dt} = 3\, m^3/sec.$



$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

when $r=2$, $\frac{dr}{dt} = \frac{3}{16\pi}$

$$\frac{dS}{dt} = 16\pi \cdot \frac{dr}{dt} = \frac{16\pi(3)}{16\pi} = 3\, m^2/sec.$$

Scratch Work