MTH 161

Midterm 1

February 15, 2022

Name: Key

UR ID: ______________________

Circle your instructor's Name:

Joshua Sumpter   Tritium Shen

Instructions

- The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.
- For each problem, please circle or box your final answer. We will judge your work outside the box as well (unless specified otherwise) so you still need to show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown, insufficient justification is given, or we are unable to read what you have written.
- In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$. However, known values of functions should be evaluated, for example, $\ln(e), \sin(\pi), e^0$.
- This exam is out of 100 points. You are responsible for checking that this exam has all 12 pages

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE: ______________________

Trig Identities

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $1 + \tan^2(\theta) = \sec^2(\theta)$
- $1 + \cot^2(\theta) = \csc^2(\theta)$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$
- $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$
- $\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
- $\sin(a)\sin(b) = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$
- $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
- $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

1. (12 points) Answer each part below and fully justify your answers.

(a) Find the equation of a line perpendicular to $y = 4x - 7$ that passes through the point $(4, 2)$.

$m = -\frac{1}{4}$

$y - 2 = -\frac{1}{4}(x - 4)$

$\boxed{y = -\frac{1}{4}x + 3}$

(b) Find an integer A such that $A = \frac{4}{3}\log_3(27) - 2\log_5(20) + \log_5(16)$.

$= \frac{4}{3}\log_3(3^3) - 2\log_5(5 \cdot 4) + \log_5(4^2)$

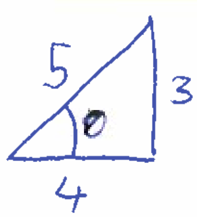
$= \left(\frac{4}{3}\right)(3) - 2\log_5(5) - \cancel{2\log_5(4)} + \cancel{2\log_5(4)}$

$= 4 - 2$

$= 2$

$\boxed{A = 2}$

(c) Find the exact value of $\cos\left(\tan^{-1}(3/4)\right)$.



$\theta = \tan^{-1}(3/4)$

$\cos(\tan^{-1}(3/4)) = \cos(\theta)$

$= \boxed{\frac{4}{5}}$

2. (12 points) Given that $f(x) = \ln(2x)$ and $g(x) = \frac{1}{2x}$, find each of the following functions and state their domain. Express your answers in terms of intervals.

(a) $(f+g)(x)$

$= f(x) + g(x) = \ln(2x) + \frac{1}{2x}$

$x \neq 0$ & $2x > 0$

Domain: $(0, +\infty)$

(b) $(f \circ g)(x)$

$= f(g(x)) = \ln\left(\frac{2}{2x}\right) = \ln\left(\frac{1}{x}\right)$

$\frac{1}{x} > 0 \Rightarrow x > 0$

Domain: $(0, +\infty)$

(c) $(g \circ f)(x)$

$= g(f(x)) = \frac{1}{2\ln(2x)}$

$\ln(2x) \neq 0$ $\qquad$ $2x > 0$

$x \neq \frac{1}{2}$ $\qquad$ $x > 0$

$(0, \frac{1}{2}) \cup (\frac{1}{2}, +\infty)$

3. (12 points) Let $f(x) = \dfrac{1}{1+x^2}$ for $\boxed{x \geq 0.}$ Answer each of the following. **Fully justify your answers.**

(a) What is the domain of f?

$1 + x^2 > 0$ for all x

$\boxed{\text{Domain: } [0, +\infty)}$

(b) Find an explicit formula for the inverse of f.

$x = \dfrac{1}{1+y^2}$

$\dfrac{1}{x} = 1 + y^2$

$y = \sqrt{\dfrac{1}{x} - 1}$

$\boxed{f^{-1}(x) = \sqrt{\frac{1}{x} - 1}}$

(c) What is the domain of $f^{-1}(x)$? Possible Hint: What is the range of f(x)?

$\dfrac{1}{x} - 1 \geq 0$

$\dfrac{1}{x} \geq 1$

$0 < x \leq 1$

$\boxed{\text{Domain: } (0, 1] = \text{Range}(f)}$

4. (30 points) Evaluate the following limits if they exist. If they do not exist, explain why not. If the limit is $+\infty$ or $-\infty$, state which it is. **Your answers must be fully justified to earn credit.**

(a) $\lim_{x\to 5} \frac{\frac{1}{5}-\frac{1}{x}}{x-5}$

$$= \lim_{x\to 5} \left(\frac{x-5}{5x}\right)\cdot\left(\frac{1}{x-5}\right)$$

$$= \lim_{x\to 5} \frac{1}{5x} = \boxed{\frac{1}{25}}$$

(c) $\lim_{x\to 3^-} \frac{x^2-2x-3}{|x-3|}$

When $x<3$, $|x-3| = 3-x$

$$= \lim_{x\to 3^-} \frac{(x-3)(x+1)}{(3-x)}$$

$$= \lim_{x\to 3^-} -(x+1) = \boxed{-4}$$

(b) $\lim_{x\to 1} \ln(\cos(1-x))$ Hint: Properties of Continuous Functions.

$\cos(1-x)$ is continuous everywhere

$\lim_{x\to 1} \cos(1-x) = \cos(0) = 1$

$\ln(x)$ is continuous at $x=1$

$$\lim_{x\to 1} \ln(\cos(\cancel{1}-x)) = \ln\left(\lim_{x\to 1} \cos(1-x)\right)$$

$$= \ln(1) = \boxed{0}$$

(d) $\lim_{x\to 2} (x^2 - 4x + 4)\cos\left(\frac{1}{x^2 - 2x}\right)$ (Hint: Use Squeeze Theorem.)

$$(x^2-4x+4) = (x-2)^2$$

$$-1 \le \cos\left(\frac{1}{x^2-2x}\right) \le 1$$

$$-(x-2)^2 \le (x-2)^2\cos\left(\frac{1}{x^2-2x}\right) \le (x-2)^2$$

$$\lim_{x\to 2} -(x-2)^2 = \lim_{x\to 2}(x-2)^2 = 0$$

By squeeze Theorem: $\lim_{x\to 2}(x-2)^2\cos\left(\frac{1}{x^2-2x}\right) = \boxed{0}$

(e) $\lim_{x\to\infty} \frac{\sqrt{x^2+4x} - \sqrt{4x}.}{x}$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2+4x}}{x} + \frac{\sqrt{4x}}{x}$$

$$= \lim_{x\to\infty} \sqrt{1+\frac{4}{x}} + \sqrt{\frac{4}{x}}$$

$$= \lim_{x\to\infty}\sqrt{1+4/x} + \lim_{x\to\infty}\sqrt{\frac{4}{x}}$$

$$= 1 + 0 = \boxed{1}$$

5. (12 points) Consider the following function:

$$f(x) = \begin{cases} -2-x & \text{if } x < -1 \\ A & \text{if } x = -1 \\ x^2+2x & \text{if } -1 < x < 0 \\ B & \text{if } x = 0 \\ \sqrt{x}+1 & \text{if } x > 0 \end{cases}$$

(a) Compute each of the following or state why they do not exist.

$\lim\limits_{x\to -1^-} f(x) = -1$ $\lim\limits_{x\to -1^+} f(x) = -1$ $\lim\limits_{x\to -1} f(x) = 1$

$\lim\limits_{x\to 0^-} f(x) = 0$ $\lim\limits_{x\to 0^+} f(x) = 1$ $\lim\limits_{x\to 0} f(x) = \text{DNE}$

(b) Find a value A such that $f(x)$ is continuous at [redacted] $x=-1$ or explain why no such value exists. Similarly, find a value B such that $f(x)$ is continuous at [redacted] $x=0$ or explain why no such value exists. **Fully justify your answer.**

$$A = \lim_{x\to -1} f(x) = 1$$

Since $\lim\limits_{x\to 0} f(x)$ DNE, there is no value B that will make f continuous at $x=0$.

6. (10 points) Let $f(x) = \ln(2x+1) + \dfrac{2x^2}{1+x}$. Use the Intermediate Value Theorem to prove that the question $f(x) = \dfrac{1}{2}$ has a solution in the interval $[0, 1]$. **Make sure to check each of the conditions of the Intermediate Value Theorem!** Hint: $\ln(3) > 0$.

- $\ln(2x+1)$ is continuous wherever $2x+1>0$, so $(-\frac{1}{2}, +\infty)$
- $\dfrac{2x^2}{1+x}$ is continuous wherever $x \neq -1$
- $f(x)$ is continuous on $[0,1]$.

$f(0) = \ln(1) + 0 = 0 < \frac{1}{2}$

$f(1) = \ln(3) + 1 > \frac{1}{2}$

Since $f(0) < \frac{1}{2} < f(1)$, there must be some x s.t. $0 < x < 1$ and $f(x) = \frac{1}{2}$.

7. (12 points) Let

$$f(x) = \begin{cases} 2^x & x \le 0 \\ \tan(x) + 1 & 0 < x < \pi/2 \\ \frac{\pi}{2x} & x \ge \pi/2 \end{cases}.$$

(a) Determine whether or not f has a vertical asymptote. If f has a vertical asymptote, state what it is. **Justify your answer using limits.**

- $\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \tan(x) + 1 = +\infty$
- f is continuous at all $x \neq \pi/2$
- f has the vertical asymptote $x = \frac{\pi}{2}$.

(b) Determine whether or not f has a horizontal asymptote. If f has a horizontal asymptote, state what it is. **Justify your answer using limits.**

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\pi}{2x} = 0$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} 2^x = 0$$

$y = 0$ is a horizontal asymptote (the only one) for $f(x)$.