MTH 161

Final

May 2, 2022

Name: Solutions

UR ID: ______________________________

Circle your instructor's Name:

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Instructions

- The presence of calculators, cell phones, bread makers, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.
- For each problem, please circle or box your final answer. We will judge your work outside the box as well (unless specified otherwise) so you still need to show your work and justify your answers. **You may not receive full credit for a correct answer if insufficient work is shown, insufficient justification is given, or we are unable to read what you have written.**
- In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2-4^2}$. However, known values of functions should be evaluated, for example, $\ln(e), \sin(\pi), e^0$.
- You must complete both parts A and B. Each part is out of 100 points. You are responsible for checking that this exam has all 18 pages, including this one and the **two pages at the end for scratch work.**

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I have read the above instructions, that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

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YOUR SIGNATURE: ______________________________

Some Volume Formulas

- Sphere: $V = \frac{4\pi}{3}r^3$.
- Circular Cone: $V = \pi r^2 \frac{h}{3}$.
- Cylinder: $V = \pi r^2 h$.
- Cube: $V = s^3$.
- Rectangular Prism (box): $V = lwh$.

Trig Identities

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $1 + \tan^2(\theta) = \sec^2(\theta)$
- $1 + \cot^2(\theta) = \csc^2(\theta)$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$
- $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$
- $\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
- $\sin(a)\sin(b) = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$
- $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
- $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

Sum Identities

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

PART A

1. Solve the following problems. **Show your work!**

   (a) **(5 points)** Solve the equation $\dfrac{5}{1+e^{-x}} = 4$ and determine whether the solution is positive or negative.

$$5 = 4 + 4e^{-x}$$

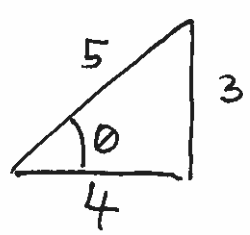
$$1 = 4e^{-x}$$

$$\frac{1}{4} = e^{-x}$$

$$\ln(1/4) = -x$$

$$\boxed{\begin{array}{l} \ln(4) = x \\ 4 > 1, \text{ so } \ln(4) > 0 \end{array}}$$

   (b) **(5 points)** Compute the exact value of $\tan(\arcsin(3/5))$.



$$\theta = \arcsin(3/5)$$

$$\tan(\theta) = \frac{3}{4}$$

2. **(18 points)** Compute the following limits using any method that you choose. If the limit is $+\infty$ or $-\infty$, determine which one. Justification using limit laws/properties/theorems is required. **Make sure to show all your work.**

(a) $\lim_{x\to\infty} \dfrac{4x^6+3x^2+2x+4}{2x^6+4x^5+x^3+1}$

$$= \lim_{x\to\infty} \frac{4+\frac{3}{x^4}+\frac{2}{x^5}+\frac{4}{x^6}}{2+\frac{4}{x}+\frac{1}{x^3}+\frac{1}{x^6}}$$

$$= \lim_{x\to\infty} \frac{4}{2} = \boxed{2}$$

(b) $\lim_{x\to\infty} \dfrac{\sin(x)}{x}$ $\qquad -1 \le \sin(x) \le 1$

$$\Rightarrow 0 = \lim_{x\to\infty} \frac{-1}{x} \le \lim_{x\to\infty} \frac{\sin(x)}{x} \le \lim_{x\to\infty} \frac{1}{x} = 0$$

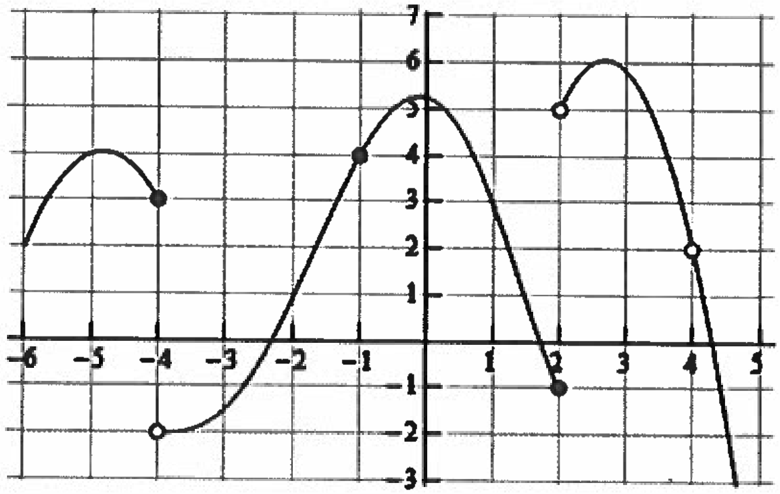
$$\Rightarrow \lim_{x\to\infty} \frac{\sin(x)}{x} = \boxed{0}$$

(c) $\lim_{x\to\infty} \ln(x^2-2) - \ln(x^4-4)$ $= \lim_{x\to\infty} \ln\left(\dfrac{x^2-2}{x^4-4}\right)$

$$\lim_{x\to\infty} \frac{x^2-2}{x^4-4} = \lim_{x\to\infty} \frac{1-\frac{2}{x^2}}{x^2-4/x^2} = 0$$

$$\Rightarrow \lim_{x\to\infty} \ln\left(\frac{x^2-2}{x^4-4}\right) = \boxed{-\infty}$$

3. The graph of a function f is given below.



(a) **(6 points)** Compute each of the following limits or state why they do not exist. If they do not exist, write 'DNE'.

$\lim_{x\to -4^-} f(x) = 3$ $\lim_{x\to -1^+} f(x) = 4$ $\lim_{x\to 2} f(x) = \text{DNE}$

$\lim_{x\to -4^+} f(x) = -2$ $\lim_{x\to -1} f(x) = 4$ $\lim_{x\to 4} f(x) = 2$

(b) **(6 points)** Based on the graph above, find any discontinuity points for f on the interval $[-6, 4.5]$. **Justify your answer using limits.**

$x=-4$ $\lim_{x\to -4^-} f(x) = 3 \neq \lim_{x\to -4^+} f(x) = -2$

$x=2$ $\lim_{x\to 2^-} f(x) = -1 \neq \lim_{x\to 2^+} f(x) = 5$

$x=4$ $f(x)$ not defined at 4.

4. Let $f(x) = 10+\sqrt{x^2+2}$ and recall the definition of the derivative $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

(a) **(10 points)** Use the definition of the derivative to find $f'(x)$. No points will be awarded if you do not use the definition.

$$f'(x) = \lim_{h\to 0} \frac{10+\sqrt{(x+h)^2+2} - (10+\sqrt{x^2+2})}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x^2+2xh+h^2+2} - \sqrt{x^2+2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2\cancel{+2}\cancel{-x^2}\cancel{-2}}{h(\sqrt{x^2+2xh+h^2+2}+\sqrt{x^2+2})}$$

$$= \lim_{h\to 0} \frac{2x+h}{(\sqrt{x^2+2xh+h^2+2}+\sqrt{x^2+2})}$$

$$= \frac{2x}{2\sqrt{x^2+2}} = \boxed{\frac{x}{\sqrt{x^2+2}}}$$

(b) **(5 points)** Find the equation of the line tangent to the graph of $f(x)$ at $x = \sqrt{2}$.

$$f'(\sqrt{2}) = \frac{\sqrt{2}}{2} \qquad f(\sqrt{2}) = 10+2 = 12$$

$$y - 12 = \frac{\sqrt{2}}{2}(x-\sqrt{2})$$

5. Consider the curve defined by $x^2y^2 = (x+1)^2(4-y^2)$

(a) **(8 points)** Find $\frac{dy}{dx}$ in terms of x and y.

$$2xy^2 + 2x^2 y\, y' = 2(x+1)(4-y^2) - 2y(x+1)^2 y'$$

$$2x^2 y\, y' + 2y(x+1)^2 y' = 2(x+1)(4-y^2) - 2xy^2$$

$$\boxed{y' = \frac{2(x+1)(4-y^2) - 2xy^2}{2x^2y + 2y(x+1)^2}}$$

(b) **(5 points)** Find the equation of all the lines tangent to the curve when $x = 0$. Hint: There are two such tangent lines.

When $x = 0$

$$0 = (0+1)^2(4-y^2)$$

$$y^2 = 4$$

$$y = \pm 2$$

$(0, 2)$: $y' = 0$

$$\boxed{y = 2}$$

$(0, -2)$: $y' = 0$

$$\boxed{y = -2}$$

6. **(20 points)** Differentiate each of the following functions. **Show all of your work!**

(a) $f(x) = \pi + \dfrac{1}{5x+1}$

$$f'(x) = 0 + \frac{d}{dx}(5x+1)^{-1}$$

$$= -5(5x+1)^{-2} = \frac{-5}{(5x+1)^2}$$

(b) $g(x) = \cancel{\cos(x)}\sin(x)\tan(x)\cancel{\sec(x)}$

$$= \sin(x)\tan(x)$$

$$g'(x) = \cos(x)\tan(x) + \sin(x)\sec^2(x)$$

(c) $h(x) = \dfrac{1+\sin(x^2)}{e^x+1}$

$f(x) = 1+\sin(x^2)$ $\quad f'(x) = 2x\cos(x^2)$

$g(x) = e^x+1$ $\quad g'(x) = e^x$

$$h'(x) = \frac{(e^x+1)(2x\cos(x^2)) - e^x(1+\sin(x^2))}{(e^x+1)^2}$$

(d) $k(x) = \cos(\tan(2e^{5x}))$

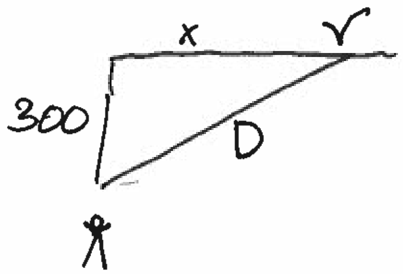
$U = 2e^{5x}$ $\quad V = \tan(u)$

$K(x) = \cos(V)$

$$K' = \frac{dK}{dV} \cdot \frac{dV}{du} \cdot \frac{du}{dx} = -\sin(V)\sec^2(u)\,10e^{5x}$$

$$= -\sin(\tan(2e^{5x}))\sec^2(2e^{5x})\,10e^{5x}$$

7. **(12 points)** A bird is flying horizontally at a constant speed of 50 ft/s. It passes over a photographer at an altitude of 300ft. At what rate is the distance between the photographer and the bird increasing 30 seconds after the bird passes over the photographer? You do not need to simplify your answer.



$x' = 50 \qquad x = 30(50) = 1500$

$D = \sqrt{(300)^2 + (1500)^2}$

$$D^2 = x^2 + (300)^2$$

$$2DD' = 2xx'$$

$$D' = \frac{2xx'}{2D} = \frac{xx'}{D} = \boxed{\frac{1500(50)}{\sqrt{(300)^2+(1500)^2}} \text{ ft/s}}$$

PART B

1. **(20 points)** Let $f(x) = (x^2 - 4)^2$ for $-1 \le x \le 3$.

   (a) Determine where $f(x)$ is increasing and where it is decreasing on $[-1, 3]$. State your answer in terms of intervals **Justify your answer!**

   $f'(x) = 4x(x^2-4) = 4x(x-2)(x+2)$

   $f'(x)$: $0 \quad + \quad 0 \quad - \quad 0 \quad +$

   $-2 \quad -1 \quad 0 \quad 2 \quad 3$

   Increasing: $[-1, 0) \cup (2, 3)$

   Decreasing: $(0, 2)$

   (b) Determine where $f(x)$ is concave up and where it is concave down on $[-1, 3]$. State your answer in terms of intervals **Justify your answer! Hint:** $2/\sqrt{3} > 1$.

   $f''(x) = 4(x^2-4) + 8x^2 = 12x^2 - 16$

   $f''(x) = 0$ when $12x^2 = 16$ or $x = \pm \frac{4}{\sqrt{12}} = \pm \frac{2}{\sqrt{3}}$

   $f''(x)$: $- \quad 0 \quad +$

   $-1 \quad 0 \quad 2/\sqrt{3} \quad 2 \quad 3$

   concave up $(2/\sqrt{3}, 3)$ concave down: $[-1, 2/\sqrt{3}]$

   (c) Find all of the critical numbers/points for $f(x)$ on $[-1, 3]$ and determine whether f has a local maximum, local minimum, or inflection point at each of the critical numbers/points. **Justify your answer!**

   critical number $\Rightarrow f'(x) = 0$

   $x = 0$ local maximum $f''(0) < 0$

   $x = 2$ local minimum $f''(2) > 0$

   (d) Find the absolute maximum and absolute minimum for $f(x)$ on $[-1, 3]$. **Justify your answer!**

   $f(1) = 9$ $\quad f(0) = 16$

   $f(2) = 0$ $\quad f(3) = 25$

   absolute minimum: 0

   absolute maximum: 25

2. Answer each part below and show all work.

(a) **(5 points)** Compute $g'(x)$ where

$$g(x) = \int_{x+1}^{x^2+4} e^{t^2+\sin(t)}\,dt.$$

Hint: Start by assuming that $F(t)$ is an antiderivative of $e^{t^2+\sin(t)}$ and use the Fundamental Theorem of Calculus (FTC).

Let $F'(t) = e^{t^2+\sin(t)}$

$g(x) = F(x^2+4) - F(x+1)$ by (FTC)

$g'(x) = 2x\, F'(x^2+4) - F(x+1)$

$= 2x\, e^{(x^2+4)^2+\sin(x^2+4)} - e^{(x+1)^2+\sin(x+1)}$ (Chain Ru

(b) **(5 points) (True or False)** Decide whether the following statement is true or false. **Fully justify your answer.**

"Let $f(x) = \frac{|x|}{x}$. $f(-2) = -1$ and $f(2) = 1$. By the Mean Value Theorem, there is a number $-2 < c < 2$ s.t. $f'(c) = \frac{1-(-1)}{2-(-2)} = \frac{1}{2}$."

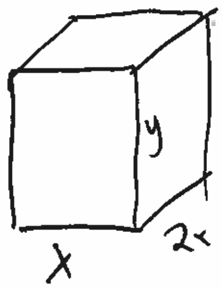
False. The MVT requires that f(x) is differentiable on (-2, 2).

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

f(x) is not continuous on (-2, 2), so not differentiabl

In particular, $f'(x) = 0$ for $x \neq 0$.

3. **(12 points)** A rectangular storage container without a lid is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the least expensive such container.



$$10 = 2x^2 y$$

$$y = \frac{5}{x^2}$$

$$C = 2x^2(10) + 6(4xy + 2xy$$

$$= 20x^2 + 36xy$$

$$= 20x^2 + \frac{180}{x}$$

$$c' = 40x - \frac{180}{x^2}$$

$$c' = 0 \iff 40x = \frac{180}{x^2}$$

$$x^3 = \frac{9}{2}$$

$$x = \sqrt[3]{9/2}$$

Minimum cost:

$$c(\sqrt[3]{9/2}) = \$\ 20(9/2)^{2/3} + 180(9/2)^{-1/3}$$

4. **(18 points)** Find the following limits or show they do not exist. If the limit is $+\infty$ or $-\infty$, determine which one. Justification is required, **show all work.**

(a) $\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{1}{2}x^2}{x^2}$ type $\frac{0}{0}$

$= \lim_{x\to 0} \frac{-\sin(x) + x}{2x}$ type $\frac{0}{0}$

$= \lim_{x\to 0} \frac{-\cos(x)+1}{2} = \frac{-1+1}{2} = \boxed{0}$

(b) $\lim_{x\to\infty} x - \ln(x)$ = type $\infty - \infty$

$= \lim_{x\to\infty} x\left(1 - \frac{\ln(x)}{x}\right)$

$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$

Since $x \to \infty$ and $\left(1 - \frac{\ln(x)}{x}\right) \to 1$

$\lim_{x\to\infty} x - \ln(x) = \infty$

(c) $\lim_{x\to\infty} (e^x + 10x)^{\frac{1}{x}}$ type ∞^0

$\lim_{x\to\infty} \ln(e^x+10x)^{1/x} = \lim_{x\to\infty} \frac{\ln(e^x+10x)}{x}$ type $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{e^x+10}{e^x+10x}$ type $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{e^x}{e^x+10}$ type $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \Rightarrow \boxed{\lim_{x\to\infty} \ln(e^x+10x)^{1/x} = e}$

5. For this problem, consider the integral $\int_0^1 x^2 + 2x\, dx$

(a) **(10 points)** Let R_n be the Riemann sum using right endpoints and n sub-intervals. Find a formula for R_n in terms of n only (no summation symbols Σ should remain in your answer). Hint: You may need some of the formulas on the formula sheet.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n} \qquad x_i = 0 + \Delta x\, i = \frac{i}{n}$$

$$R_n = \sum_{i=1}^{n} \left[(i/n)^2 + (2i/n)\right](1/n)$$

$$= \left(\frac{1}{n}\right) \sum_{i=1}^{n} \left(\frac{i^2}{n^2}\right) + \left(\frac{2}{n}\right) i$$

$$= \frac{1}{n}\left[\frac{1}{n^2}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} i\right]$$

$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right)$$

$$R_n = (n+1)(2n+1)/(6n^2) + \frac{n+1}{n}$$

(b) **(5 points)** Calculate $\lim_{n\to\infty} R_n$ to find the exact value of the definite integral.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} + \frac{(n+1)}{n}$$

$$= \frac{2}{6} + \frac{1}{1} = \boxed{\frac{1}{3} + 1}$$

(b) $\int \sin^5(x)\cos^3(x)\,dx$

Hint: You will want to use the identity $\sin^2(x)+\cos^2(x)=1$.

$u = \sin(x) \qquad \frac{du}{\cos(x)} = dx$

$= \int u^5 \cos^2(x)\,du = \int u^5(1-u^2)\,du$

$= \int u^5 - u^7\,du = \frac{1}{6}u^6 - \frac{1}{8}u^8 + C$

$= \frac{1}{6}\sin^6(x) - \frac{1}{8}\sin^8(x) + C$

(c) $\int_1^2 \frac{x^2-1}{x^2+x}\,dx$

$= \int_1^2 \frac{(x+1)(x-1)}{x(x+1)}\,dx = \int_1^2 \frac{x-1}{x}\,dx = \int_1^2 1 - \frac{1}{x}\,dx$

$= x - \ln|x| \Big|_{x=1}^{x=2} = (2-\ln(2)) - (1-\cancel{\ln(1)}^{=0})$

$= 1 - \ln(2)$

(d) $\int_1^e \frac{\ln(x)}{2x}\,dx$

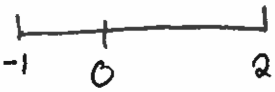
$u = \ln(x) \qquad x\,du = dx$

$= \int_0^1 \frac{u}{2x}\cdot x\,du = \frac{1}{2}\int_0^1 u\,du$

$= \frac{1}{2}\left(\frac{u^2}{2}\Big|_{u=0}^{u=1}\right)$

$= \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

6. **(5 points)** If $\int_{-1}^{2} f(x)\,dx = 3$ and $\int_{0}^{2} f(x)\,dx = -4$, find $\int_{0}^{-1} 5f(x) + 2\,dx$.



$$\int_{-1}^{0} f(x)dx = \int_{-1}^{2} f(x)dx - \int_{0}^{2} f(x)dx$$

$$= 3-(-4) = 7$$

$$\int_{0}^{-1} 5f(x)+2dx = -5\int_{-1}^{0} f(x)dx + \int_{0}^{-1} 2\,dx$$

$$= -35 - 2 = \boxed{-37}$$

7. **(20 points)** Evaluate each of the following integrals.

(a) $\int x^3 + 3x^2 - 2x + 1\,dx$

$$\frac{1}{4}x^4 + x^3 - x^2 + x + C$$

(continued on next page →)