MTH 161

Midterm 1

Thursday, February 22, 2018

NAME (please print legibly): KEY

Your University ID Number: ________________

Circle your instructor and class time:

Lorman (MW 2:00) Peng (MW 4:50)

Please read the following instructions very carefully:

- You have **75 minutes** to complete this exam.
- Only pens/pencils are allowed. The presence of notes, calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden. You may refer to the **formula sheet** on the last page of this exam.
- Show your work and justify your answers. If you need extra space, use the back of the previous page and **clearly indicate** that you have done so. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given. **Clearly circle or label your final answers.**
- Sign the following academic honesty statement: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.*

Signature: ________________

QUESTION	VALUE	SCORE
1	20	
2	15	
3	25	
4	15	
5	10	
6	15	
TOTAL	100	

1. (20 points)

(a) Find all solutions to the equation

$$2^{1-x^2} = 1.$$

$$\log_2(2^{1-x^2}) = \log_2(1)$$

$$1 - x^2 = 0$$

$$x^2 = 1$$

$$\boxed{x = \pm 1}$$

(b) Find all solutions to the equation

$$|x| = |x+1|.$$

$x = 0$ not a solution, so we can divide by x.

$$\frac{|x+1|}{|x|} = 1$$

$$\left|\frac{x+1}{x}\right| = 1$$

$$\frac{x+1}{x} = 1 \quad \text{or} \quad \frac{x+1}{x} = -1$$

$$x+1 = x \qquad\qquad x+1 = -x$$

$$\cancel{1 = 0} \qquad\qquad 2x+1 = 0$$

$$\boxed{x = -\frac{1}{2}}$$

(c) Find the solution set of the following inequality:

$$|x-3| > 4.$$

$x - 3 > 4$ or $x - 3 < -4$

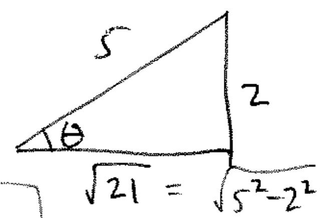
$x > 7$ or $x < -1$

$(-\infty, -1) \cup (7, \infty)$

(d) Find the value of $\cos(\sin^{-1}(\frac{2}{5}))$.

Let $\theta = \sin^{-1}\left(\frac{2}{5}\right)$

$\sin(\theta) = \frac{2}{5}$



$\boxed{\cos\theta = \cos\left(\sin^{-1}\left(\frac{2}{5}\right)\right) = \frac{\sqrt{21}}{5}}$

(e) Find all solutions to the following equation in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$:

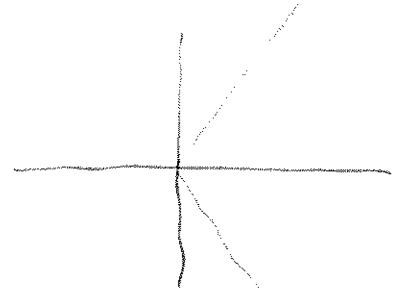
$$\cos(2x) = -\frac{1}{2}.$$

$\cos(2x) = 2\cos^2 x - 1 = -\frac{1}{2}$

$2\cos^2 x = \frac{1}{2}$

$\cos^2 x = \frac{1}{4}$

$\cos x = \pm 1/2$



Solutions between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$: $\boxed{x = -\frac{\pi}{3}, \frac{\pi}{3}}$

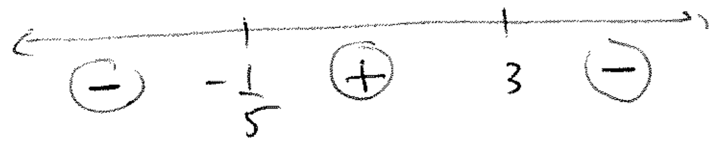
2. (15 points) Let

$$f(x) = \sqrt{\frac{1+5x}{3-x}}$$

(a) Find the domain of $f(x)$.

Need $3 - x \neq 0$ and $\frac{1+5x}{3-x} \geq 0$

$x \neq 3$



Plug in and check +/−.

Domain: $\left[-\frac{1}{5}, 3\right)$

(b) Find $f^{-1}(x)$.

$$x = \sqrt{\frac{1+5y}{3-y}}$$

$$x^2 = \frac{1+5y}{3-y}$$

$$3x^2 - yx^2 = 1+5y$$

$$-yx^2 - 5y = 1 - 3x^2$$

$$y(-x^2-5) = 1 - 3x^2$$

$$f^{-1}(x) = y = \frac{1-3x^2}{-x^2-5}$$

or

$$\boxed{f^{-1}(x) = y = \frac{3x^2-1}{x^2+5}}$$

3. (25 points) Find each of the following limits or show they do not exist. (If the limit approaches ∞ or $-\infty$, specify which one.)

(a)

$$\lim_{x \to 1} \ln x$$

$= \ln(1) = 0$

since $\ln x$ is continuous at $a=1$.

(b)

$$\lim_{t \to 4} \frac{4-t}{2-\sqrt{t}}$$

$= \lim_{t \to 4} \frac{(4-t)}{(2-\sqrt{t})} \cdot \frac{(2+\sqrt{t})}{(2+\sqrt{t})} = \lim_{t \to 4} \frac{(4-t)(2+\sqrt{t})}{4-t}$

$= \lim_{t \to 4} 2+\sqrt{t} = \boxed{4}$

(c)

$$\lim_{x\to 1^-} \frac{3x-2}{1-|x|}$$

When x is near 1, $|x| = x$

so $$\lim_{x\to 1^-} \frac{3x-2}{1-|x|} = \lim_{x\to 1^-} \frac{3x-2}{1-x}$$

} approaches 1

} approaches 0 from the positive side

$$= +\infty$$

(d)

$$\lim_{x\to 1^-} \frac{3x-2-|x-2|}{1-|x|}$$

When x is near 1, $|x| = x$

$x-2<0$ so $|x-2| = -(x-2)$

and $$\lim_{x\to 1^-} \frac{3x-2-|x-2|}{1-|x|} = \lim_{x\to 1^-} \frac{3x-2+(x-2)}{1-x}$$

$$= \lim_{x\to 1^-} \frac{4x-4}{1-x} = \lim_{x\to 1^-} \frac{-4(1-x)}{1-x} = \boxed{-4}$$

(e)

$$\lim_{h\to 0} \frac{\frac{1}{(h+3)^2} - \frac{1}{9}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{9-(h+3)^2}{9(h+3)^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{9} - (h^2+6h+\cancel{9})}{9h(h+3)^2}$$

$$= \lim_{h\to 0} \frac{-h^2-6h}{9h(h+3)^2} = \lim_{h\to 0} \frac{\cancel{h}(-h-6)}{9\cancel{h}(h+3)^2}$$

$$= \lim_{h\to 0} \frac{-h-6}{9(h+3)^2} = \frac{-6}{9(3^2)} = \frac{-6}{81} = \boxed{\frac{-2}{27}}$$

4. (15 points)

(a) State (precisely) what it means for a function $f(x)$ to be *continous* at a number a.

$f(x)$ is continuous at a if

$$\lim_{x\to a} f(x) = f(a)$$

(b) Let

$$f(x) = \begin{cases} 2\sin(\frac{\pi x}{4}) & x \le -1 \\ |x| & -1 < x < 1 \\ e^{x-1} & x \ge 1 \end{cases}$$

Find the number(s) a at which f is NOT continuous and give the reason(s).

$2\sin\left(\frac{\pi x}{4}\right)$, $|x|$, and e^{x-1} are all continuous in their domains so the only possible discontinuities are at $a = \pm 1$.

When $a=-1$:

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 2\sin\left(\frac{\pi x}{4}\right) = 2\sin\left(-\frac{\pi}{4}\right) = 2\cdot\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} |x| = |-1| = 1 .$$

$\lim_{x\to -1^-} \neq \lim_{x\to -1^+}$ so $\lim_{x\to -1} f(x)$ DNE. $\boxed{f \text{ is } \underline{\text{not}} \text{ continuous at } a=-1}$

When $a=1$: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} |x| = 1$. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} e^{x-1} = e^0 = 1$

and $f(1) = e^0 = 1$.

So f is continuous at $a=1$.

5. (10 points) Consider the equation

$$\frac{x^5+3x+7}{x^3+x}=0$$

(a) Use the Intermediate Value Theorem to show that there is a solution to this equation between $a=-2$ and $b=-1$. (Showing your work means checking that the conditions necessary to apply the Intermediate Value Theorem apply.)

$f(x) = \frac{x^5+3x+7}{x^3+x} = \frac{x^5+3x+7}{x(x^2+1)}$ has domain

all x ~~#~~ except $x=0$. f is continuous

on $[-2,-1]$. $f(-2) = \frac{-32-6+7}{-8-2} = \frac{-31}{-10} = \frac{31}{10} > 0$

$f(-1) = \frac{-1-3+7}{-1-1} = \frac{3}{-2} < 0$

So by the IVT, $f(x)=0$ has a solution between -2 and -1.

(b) Does the Intermediate Value Theorem apply to show that the equation has a solution between $a=-1$ and $b=2$? Explain why or why not.

No. f is not continuous at 0 (since undefined at 0) so it is not continuous on $[-1,2]$.

So the IVT does not apply.

6. (15 points) Consider the function

$$f(x) = \frac{1+x}{\sqrt{x+x^2}}.$$

Its domain is $(-\infty,-1) \cup (0,\infty)$.

[*Possibly useful fact*: if $x \neq 0$, then $\sqrt{x+x^2} = \sqrt{x^2(1+\frac{1}{x})}$ and $1+x = x(1+\frac{1}{x})$.]

(a) Find all vertical asymptotes of the curve $y = f(x)$.

Write $f(x) = \dfrac{1+x}{\sqrt{x+x^2}} = \dfrac{x(1+\frac{1}{x})}{\sqrt{x^2(1+\frac{1}{x})}} = \dfrac{x(1+\frac{1}{x})}{|x|\sqrt{1+\frac{1}{x}}} = \dfrac{x}{|x|}\cdot\sqrt{1+\frac{1}{x}}$.

Vertical asymptotes only possible at $x = 0, -1$.

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x}{|x|}\sqrt{1+\frac{1}{x}} = \lim_{x\to -1^-} \frac{x}{(-x)}\sqrt{1+\frac{1}{x}} = 0$$

so $\underline{\text{no}}$ vert. asymptote at -1.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \underbrace{\frac{x}{|x|}}_{\to 1}\underbrace{\sqrt{1+\frac{1}{x}}}_{\to +\infty} = +\infty$$

$\boxed{\text{Vertical asymptote at } x = 0}$

(b) Find all horizontal asymptotes of the curve $y = f(x)$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \underbrace{\frac{x}{|x|}}_{\to 1}\underbrace{\sqrt{1+\frac{1}{x}}}_{\to 1} = 1$$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \underbrace{\frac{x}{|x|}}_{\to -1}\underbrace{\sqrt{1+\frac{1}{x}}}_{\to 1} = -1$$

so $y = -1$ and $y = 1$ are horizontal asymptotes