MTH 161

Midterm 2

April 04, 2024

Name: ____

UR ID: ____

Circle your Instructor's Name:

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Instructions:

• The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.

• For each problem, please put your final answer in the answer box. We will judge your work outside the box as well (unless specified otherwise) so you still need to show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

• In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$.

• This exam is out of 100 points. You are responsible for checking that this exam has all 12 pages.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:____

Some Volume Formulas

- Sphere: $V = \frac{4\pi}{3} r^3$.
- Circular cone: $V = \pi r^2 \frac{h}{3}$
- Cylinder: $V = \pi r^2 h$.
- Cube: $V = s^3$.
- Rectangular prism (box): $V = lwh$.

Trig Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$
- $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
- $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
- $\sin(a)\cos(b) = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$
- $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
- $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

1. (18 points) Compute the following limits. Show all your work.

(a) $\lim_{x\to 0} \frac{\sin(2024x)\sin(x)}{x^2}$

$$= \left(\lim_{x\to 0} \frac{\sin(2024x)}{x}\right) \cdot \underbrace{\lim_{x\to 0} \frac{\sin x}{x}}$$

$2024x \to 0$ as $x \to 0$

$$= \left(\lim_{x\to 0} \frac{\sin(2024x)}{2024x} \cdot 2024\right) \cdot 1$$

$$= (1 \cdot 2024) \cdot 1$$

$$= \boxed{2024}$$

(b) $\lim_{h\to 0} \frac{\ln(e+h)-1}{h}$. **Hint:** The limit represents the derivative of a function f at a number a.

Let $f(x) = \ln x$, $a = e$. The derivative of $\ln x$ at e can be represented as

$$\lim_{h\to 0} \frac{\ln(e+h) - \ln(e)}{h} = \lim_{h\to 0} \frac{\ln(e+h) - 1}{h}$$

For $f(x) = \ln x$, $f'(e) = \frac{1}{e}$

Thus, $\lim_{h\to 0} \frac{\ln(e+h)-1}{h} = \boxed{\frac{1}{e}}$

(c) $\lim_{x\to\infty} \frac{(x^2+5)(3x^2+4)}{(x^2-7)^2} = \lim_{x\to\infty} \frac{(x^2+5)\,(3x^2+4)}{(x^2-7)\,(x^2-7)}$

$$= \lim_{x\to\infty} \frac{x^2+5}{x^2-7} \cdot \lim_{x\to\infty} \frac{3x^2+4}{x^2-7}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}(1+\cancel{\frac{5}{x^2}}^{\to 0})}{\cancel{x^2}(1-\cancel{\frac{7}{x^2}}^{\to 0})} \cdot \lim_{x\to\infty} \frac{\cancel{x^2}(3+\cancel{\frac{4}{x^2}}^{\to 0})}{\cancel{x^2}(1-\cancel{\frac{7}{x^2}}^{\to 0})}$$

$$= \frac{1}{1} \cdot \frac{3}{1}$$

$$= \boxed{3}$$

2. (18 points) Let $2\sin y + y\cos x = x + \pi$.

(a) Find $\frac{dy}{dx}$ by using implicit differentiation.

Differentiate both sides:

$$2\cos y \cdot y' + \cos x \cdot y' - y\sin x = 1$$

$$2\cos y \cdot y' + \cos x \cdot y' = 1 + y\sin x$$

$$y'(2\cos y + \cos x) = 1 + y\sin x$$

$$\boxed{\frac{dy}{dx} = y' = \frac{1 + y\sin x}{2\cos y + \cos x}}$$

(b) Find the equation of tangent line at the point $(0, \pi)$.

$y - \pi = m(x - 0)$, where m is y' at $(0, \pi)$

$$y' \text{ at } (0, \pi) = \frac{1 + \pi \cdot \sin(0)}{2\cos(\pi) + \cos(0)} = \frac{1+0}{-2+1}$$

$$= -1$$

$$y - \pi = -(x - 0)$$

$$\boxed{y = \pi - x}$$

3. (20 points) Find the derivative of each function below. Show all your work.

(a) $f(x) = \dfrac{(x^2 - 4x + 4)^3}{x - 2} - \sin(c)$, where c is a constant.

constant

$$f(x) = \frac{((x-2)^2)^3}{x-2} - \sin(c)$$

$$f(x) = (x-2)^5 - \sin(c)$$

$$f'(x) = 5(x-2)^4 - 0$$

$$= \boxed{5(x-2)^4}$$

(b) $g(x) = 3^{\sin x} + cos^3 x - \tan(x^3)$

$$\frac{d}{dx}(g(x)) = \frac{d}{dx}\left(3^{\sin x}\right) + \frac{d}{dx}\left(\cos^3 x\right) - \frac{d}{dx}\left(\tan(x^3)\right)$$

$$= \ln(3) \cdot 3^{\sin x} \cdot \frac{d}{dx}(\sin x) + 3\cos^2 x \cdot \frac{d}{dx}(\cos x) - \sec^2(x^3) \cdot \frac{d}{dx}(x^3)$$

$$= \boxed{\ln(3) \cdot 3^{\sin x} \cdot \cos x - 3\cos^2 x \cdot \sin x - 3\sec^2(x^3)\, x^2}$$

(c) $h(x) = x^2 \cdot \ln\left(\arcsin(x)\right)$

$$\frac{d}{dx}\left(h(x)\right) = \frac{d}{dx}\left(x^2\right) \cdot \ln\left(\arcsin(x)\right) + x^2 \cdot \frac{d}{dx}\left(\ln \arcsin(x)\right)$$

$$= 2x \cdot \ln\left(\arcsin(x)\right) + x^2 \cdot \frac{\frac{d}{dx}\left(\arcsin x\right)}{\ln\left(\arcsin(x)\right)}$$

$$= \boxed{2x \cdot \ln\left(\arcsin(x)\right)}$$

(d) $k(x) = (\sin x)^{e^x}$

$$\ln k(x) = \ln\left[(\sin x)^{e^x}\right] = e^x \cdot \ln(\sin x)$$

$$\frac{k'(x)}{k(x)} = \left(e^x \cdot \ln(\sin x)\right)'$$

$$k'(x) = k(x) \cdot \left[(e^x)' \cdot \ln(\sin x) + e^x \cdot \left(\ln(\sin x)\right)'\right]$$

$$= (\sin x)^{e^x} \cdot \left[e^x \cdot \ln(\sin x) + e^x \cdot \frac{(\sin x)'}{\sin x}\right]$$

$$= \boxed{(\sin x)^{e^x} \cdot e^x \cdot \left(\ln(\sin x) + \cot x\right)}$$

4. (15 points) A particle travels along a line with respect to time (in seconds) according the following position function:

$$s(t) = t^3 - 6t^2 + 9t \quad (t \geq 0),$$

where $s(t)$ is in terms of meters.

(a) (6pts) When is the particle moving in the negative direction?

$v(t) < 0$ iff the particle moves in the negative direction.

$v(t) = 3t^2 - 12t + 9 = 3(t-3)(t-1)$

1 3

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$v(4) = 9$

$v(2) = -3$

$v(0) = 9$

So, $v(t) < 0$ when $\boxed{1 < t < 3 \quad (1, 3)}$

(b) (6pts) Find the total distance the particle travels during the first 4 seconds?

$s(0) = 0$, $s(1) = 1^3 - 6 \cdot 1^2 + 9 \cdot 1 = 4$

$s(3) = 3^3 - 6 \cdot 3^2 + 9 \cdot 3 = 0$, $s(4) = 4^3 - 6 \cdot 4^2 + 9 \cdot 4 = 4$

Total distance $= |s(1) - s(0)| + |s(3) - s(1)| + |s(4) - s(3)|$

$|4 - 0| + |0 - 4| + |4 - 0| = \boxed{12\,m}$

(c) (3pts) Find the acceleration of the particle at $t = 4$ sec.

$a(t) = v'(t) = 6t - 12$

$a(4) = 6 \cdot 4 - 12 = \boxed{12\ m/s^2}$

5. (15 points) The base radius of a circular cone is **increasing** at a rate of 2 inches per second and the height of the cone is **decreasing** at a rate of 3 inches per second. How fast is the volume of the cone changing when the base radius is 4 inches and the height is 6 inches?

$$\boxed{\begin{gathered}\frac{dr}{dt} = 2 \text{ inch/sec} \\ \frac{dh}{dt} = -3 \text{ inch/sec} \\ r = 4 \text{ inch}, \quad h = 6 \text{ inch}\end{gathered}}$$

$$\text{Volume} = \frac{1}{3}\pi r^2 h,$$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi r^2 h\right)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi \cdot \frac{d}{dt}(r^2 h)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi \cdot \left(\frac{d}{dt}(r^2)\cdot h + r^2 \cdot \frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi \left(2r \cdot \frac{dr}{dt} \cdot h + r^2 \cdot \frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi \left(2\cdot 4 \cdot 2 \cdot 6 + 4^2 \cdot (-3)\right)$$

$$= \frac{1}{3}\pi (96 - 48) = \boxed{16\pi \text{ inch}^3/\text{sec}}$$

6. (14 points) Use the linearization of the function $f(x) = \sqrt[4]{x}$ at the point $a = 81$ to approximate $\sqrt[4]{82}$.

Let $f(x) = \sqrt[4]{x} = x^{1/4}$, so $f'(x) = \dfrac{1}{4x^{3/4}}$

Then, for $x \approx a$

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

For $x = 82$, $a = 81$

$$\sqrt[4]{82} = f(82) \approx L(82) = f(81) + f'(81)(82-81)$$

$$\sqrt[4]{82} \approx 81^{1/4} + \frac{1}{4 \cdot 81^{3/4}} \cdot 1$$

$$\sqrt[4]{82} \approx 3 + \frac{1}{108}$$

$$\boxed{\sqrt[4]{82} \approx \frac{325}{108}}$$

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