MTH 161

Midterm 1

February 22, 2024

Name: Solutions

UR ID: ______________________

Circle your Instructor's Name:

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Instructions:

- The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.
- Show your work and justify your answers! You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$.
- This exam is out of 100 points. You are responsible for checking that this exam has all 11 pages.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________________

Trig Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$
- $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
- $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
- $\sin(a)\cos(b) = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$
- $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
- $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

1. (15 points) Answer each part below and fully justify your answers:

(a) Given $\tan\theta = \frac{-4}{3}$, and $\cos\theta > 0$, compute $\sin(\theta)$.

$\tan\theta < 0$ and $\cos\theta > 0$, so $\sin\theta < 0$

$5^2 = 3^2 + 4^2$

$\sin\alpha = \frac{4}{5}$, $\sin\theta = -\sin\alpha$

$= \boxed{\frac{-4}{5}}$

(b) Find the exact value of $\cos(\sin^{-1}(3/8))$.

Let $\theta = \sin^{-1}(3/8)$. Then,

$\sin(\theta) = 3/8$

and $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, but $\frac{3}{8} > 0$, so θ is an acute angle. $\cos(\sin^{-1}(3/8)) = \cos\theta = \sqrt{1 - \left(\frac{3}{8}\right)^2}$

$= \boxed{\frac{\sqrt{55}}{8}}$

(c) A, B, and C are defined as

$$A = |x| - |-x|$$
$$B = \ln\left((\sin x)^2 + (\cos x)^2\right)$$
$$C = \frac{1}{2}\log_5(16) - \log_5(20)$$

Compute $A + B + C$. Your final answers should not contain x.

$A = 0$ as $|x| = |-x|$ for all x

$B = \ln(\sin^2 x + \cos^2 x) = \ln(1) = 0$

$C = \log_5 16^{1/2} - \log_5 20 = \log_5 4 - \log_5 20$

$= \log_5 (4/20)$

$= -1$

$\boxed{A + B + C = -1}$

2. (12 points) Complete each part below.

(a) Solve the following equation for x: $e^x - 3 = \frac{4}{e^x}$.

Let $a = e^x$. Then we get:

$$a - 3 = \frac{4}{a}, \quad a > 0 \text{ as } e^x > 0$$

For all x.

$$a^2 - 3a = 4$$

$$a^2 - 3a - 4 = 0$$

$$(a-4)\cdot(a+1) = 0, \quad a = 4 \text{ or } a = -1$$

But $a > 0$, so $a = 4$. $e^x = 4$. So, $\boxed{x = \ln 4}$

(b) Find all x values, $0 \le x \le 2\pi$ such that $\sin(2x) = \tan(x)$.

$$\sin(2x) = 2\sin x \cos x = \frac{\sin x}{\cos x}$$

$$\sin x = 0 \quad \text{or} \quad 2\cos x = \frac{1}{\cos x}$$

$$x = 0, \pi, 2\pi \qquad 2\cos^2 x = 1$$

$$\cos^2 x = \frac{1}{2}$$

$$\boxed{x = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$$

$$\cos x = \pm\sqrt{\frac{1}{2}}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

3. (15 points) Let $f(x) = log_3(4x) - log_3(5 - x)$.

(a) What is the domain of f?

$4x > 0$ and $5 - x > 0$

$x > 0$ and $5 > x$

$\boxed{0 < x < 5}$ OR $\boxed{(0, 5)}$

(b) Find an explicit formula for the inverse of f.

$y = \log_3(4x) - \log_3(5-x)$

$y = \log_3\left(\frac{4x}{5-x}\right)$, so $\frac{4x}{5-x} = 3^y$

$4x = 5 \cdot 3^y - x \cdot 3^y$

$4x + x \cdot 3^y = 5 \cdot 3^y$, so $x \cdot (4 + 3^y) = 5 \cdot 3^y$

$x = \frac{5 \cdot 3^y}{4 + 3^y}$

$\boxed{f^{-1}(x) = \frac{5 \cdot 3^x}{4 + 3^x}}$

(c) What is the domain of $f^{-1}(x)$?

$f^{-1}(x) = \frac{5 \cdot 3^x}{4 + 3^x}$

$3^x + 4 > 0$ for all x.

$\mathrm{Dom}(f^{-1}) = \boxed{(-\infty, \infty)}$

OR $\boxed{\mathbb{R}}$

4. (15 points)

Let $f(x) = \sqrt{\dfrac{1}{x+2}}$, $g(x) = x^2$.

(a) Find $g \circ f$.

$$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{\frac{1}{x+2}}\right) = \left(\sqrt{\frac{1}{x+2}}\right)^2 = \boxed{\frac{1}{x+2}}$$

(b) Find the domain of $g \circ f$.

Both $\underbrace{\sqrt{\frac{1}{x+2}}}_{f(x)}$ and $\underbrace{\frac{1}{x+2}}_{g(f(x))}$ should make sense.

So, $\boxed{x > -2}$ OR $\boxed{(-2, \infty)}$

(c) Find the range of $g \circ f$.

$(g \circ f)(x) = \dfrac{1}{x+2}$ and $x > -2$.

So, $x+2 > 0$. So, $g \circ f$ takes positive values only. Moreover, it can take all positive values, since $x+2$ can be any positive number.

Range of $g \circ f$ is $\boxed{(0, \infty)}$

5. (28 points) Evaluate the following limits if they exist. If they do not exist, explain why not. Your answers must be fully justified to earn credit.

(a) $\displaystyle\lim_{x\to 4} \frac{4-x}{2-\sqrt{x}}$

Substitution does NOT work:

$$\frac{4-4}{2-\sqrt{4}} = \text{"}\frac{0}{0}\text{"}$$

$$\lim_{x\to 4} \frac{4-x}{2-\sqrt{x}} = \lim_{x\to 4} \frac{(2+\sqrt{x})\cdot\cancel{(2-\sqrt{x})}}{\cancel{(2-\sqrt{x})}}$$

$$= \lim_{x\to 4} 2+\sqrt{x} = 2+\sqrt{4}$$

$$= \boxed{4}$$

(b) $\displaystyle\lim_{x\to 3} \frac{|3-x|\cdot(x-2)}{x^2+x-12}$

$$\lim_{x\to 3^+} \frac{|3-x|\,(x-2)}{x^2+x-12} = \lim_{x\to 3^+} \frac{\cancel{(x-3)}\cdot(x-2)}{\cancel{(x-3)}\,(x+4)}$$

$$= \frac{1}{7}$$

$$\lim_{x\to 3^-} \frac{|3-x|\cdot(x-2)}{x^2+x-12} = \lim_{x\to 3^-} \frac{(3-x)\cdot(x-2)}{(x-3)\,(x+4)} = \frac{-1}{7}$$

As $\frac{1}{7} \neq \frac{-1}{7}$, $\boxed{\text{the limit does not exist.}}$

(c) $\lim_{x\to 0} \left(\sqrt{x^2+x^4}\right)\cdot \sin\left(\frac{2024\pi^2}{x}\right)$. Hint: Use the Squeeze Theorem.

$$-1 \le \sin\left(\frac{2024\pi^2}{x}\right) \le 1 \quad \text{for all } x \neq 0$$

$$\underbrace{-\sqrt{x^2+x^4}}_{g(x)\to 0 \text{ as } x\to 0} \le \underbrace{\sqrt{x^2+x^4}\cdot \sin\left(\frac{2024\pi^2}{x}\right)}_{F(x)\to 0 \text{ as } x\to 0} \le \underbrace{\sqrt{x^2+x^4}}_{h(x)\to 0 \text{ as } x\to 0}$$

$\boxed{0}$

(d) $\lim_{x\to 2} \cos\left(\sqrt{\pi^2+\ln\left(\frac{5+x^3}{11+x}\right)}\right)$.

cos is continuous everywhere,

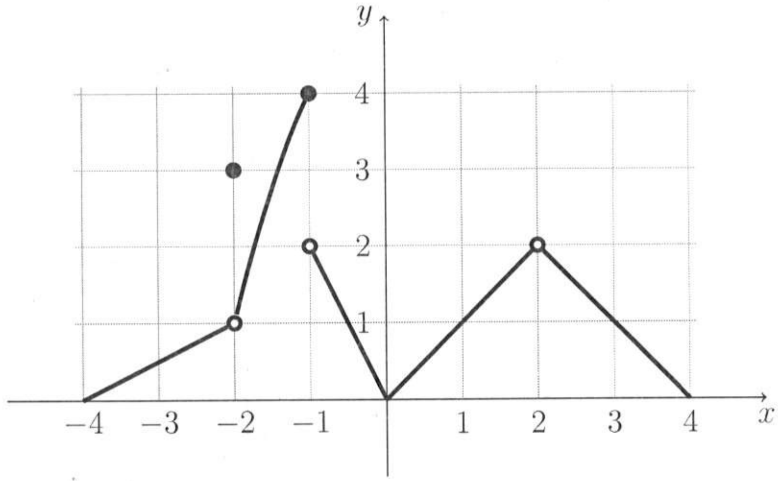
$\sqrt{\ }$ is continuous on $[0,\infty)$ and ln is cont. on $(0,\infty)$. So,

$$\lim_{x\to 2} \cos\left(\sqrt{\pi^2+\ln\left(\frac{5+x^3}{11+x}\right)}\right) = \cos\left(\sqrt{\pi^2+\lim_{x\to 2}\ln\left(\frac{5+x^3}{11+x}\right)}\right)$$

$$= \cos\left(\sqrt{\pi^2+\ln 1}\right) = \cos(\pi) = \boxed{-1}$$

6. (15 points)

The function f has the following graph:



Consider the list of numbers $-3, -2, -1, 0, 1, 2, 3$.

(a) At what number(s) a from the list does $\lim\limits_{x\to a} f(x)$ not exist?

$$\lim_{x\to -1^+} f(x) = 2 \neq 4 = \lim_{x\to -1^-} f(x)$$

So, $\boxed{\lim_{x\to -1} f(x) \text{ does not exist. } \quad a = -1}$

The limit exists at other numbers.

(b) At what number(s) a does $\lim_{x \to a} f(x)$ exist but f is not continuous at a?

$$\lim_{x \to -2} f(x) = 1 \quad \text{and} \quad \lim_{x \to 2} f(x) = 2$$

But $f(-2) \neq 1$ and $f(2)$ is undefined.

So, $\boxed{a = 2, -2}$

(c) On the domain [0,4], let $g(x)$ be defined as:

$$g(x) = \begin{cases} f(x) + A & \text{if} \quad 0 \leq x < 2 \\ B & \text{if} \quad x = 2 \\ 3f(x) & \text{if} \quad 2 < x \leq 4 \end{cases}$$

Find the constants A and B making $g(x)$ continuous on its domain.

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} f(x) + A = \left(\lim_{x \to 2^-} f(x)\right) + A$$

$$= 2 + A$$

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} 3f(x) = 3 \cdot \left(\lim_{x \to 2^+} f(x)\right)$$

$$= 6$$

$g(2) = B$.

So, $2 + A = 6 = B$. thus, $\boxed{A = 4, \ B = 6}$

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