1. (12 points) Consider the curve defined by the following equation:

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2.$$

(a) Find y'.

$$\rightarrow \frac{d}{dx}(2y^3 + y^2 - y^5) = \frac{d}{dx}(x^4 - 2x^3 + x^2)$$

$$\Rightarrow 6y^2 \frac{dy}{dx} + 2y \frac{dy}{dx} - 5y^4 \frac{dy}{dx} = 4x^3 - 6x^2 + 2x$$

$$\Rightarrow \frac{dy}{dx}(6y^2 + 2y - 5y^4) = 4x^3 - 6x^2 + 2x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{4x^3 - 6x^2 + 2x}{6y^2 + 2y - 5y^4}}$$

(b) Find the x-coordinates/values for which the curve has horizontal tangent lines.

$$\rightarrow \frac{dy}{dx} = 0 \Rightarrow 4x^3 - 6x^2 + 2x = 0$$

$$\Rightarrow 2x(2x^2 - 3x + 1) = 0$$

$$\Rightarrow 2x(2x-1)(x-1) = 0$$

$$\Rightarrow x = 0, 1, \tfrac{1}{2}.$$

2. (24 points) Compute the derivative of each of the following functions:

(a) If c is a non-zero constant and $f(x) = \dfrac{x}{x + c/x}$, find $f'(x)$.

$$\Rightarrow f'(x) = \frac{x + c/x - x(1 - c/x^2)}{(x + c/x)^2}$$

(b) Let $g(x) = \tan^{-1}(x^2)$ and find $g'(x)$. (Recall: $\tan^{-1}(x) = \arctan(x)$).

$$\Rightarrow \quad g'(x) = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$$

(c) Let $h(x) = \ln\left(\frac{\sqrt{1+2x}}{\sqrt{1-2x}}\right)$ and find $h'(x)$.

$$\Rightarrow \quad h(x) = \ln\left[(1+2x)^{1/2}\right] - \ln\left[(1-2x)^{1/2}\right]$$

$$= \frac{1}{2}\ln(1+2x) - \frac{1}{2}\ln(1-2x)$$

$$\Rightarrow \quad h'(x) = \frac{1}{2}\,\frac{1}{1+2x}(2) - \frac{1}{2}\cdot\frac{1}{1-2x}(-2)$$

$$= \frac{1}{1+2x} + \frac{1}{1-2x}$$

(d) Let $p(x) = (x^2\sin(x))^x$ and find $p'(x)$.

Let $y = (x^2\sin x)^x \Rightarrow \ln y = \ln\left[(x^2\sin x)^x\right]$

$$= x\ln(x^2\sin x)$$

$$= x\left(\ln(x^2) + \ln(\sin x)\right)$$

$$= x\left(2\ln x + \ln(\sin x)\right)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 1\cdot\left(2\ln x + \ln(\sin x)\right) + x\left(\frac{2}{x} + \frac{1}{\sin x}\cos x\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[2\ln x + \ln(\sin x) + 2 + x\cot x\right]$$

$dx \quad 0 \quad L$

$$= (x^2 \sin x)^x \left[2\ln x + \ln \sin x + 2 + x \cot x\right]$$

3. (15 points) Let

$$s(t) = t^2(1-t) \qquad 0 \le t \le 1$$

denote the position of a particle in meters at time t in seconds.

(a) Compute the velocity $v(t)$ of the particle at time t.

$\rightarrow \quad s = t^2 - t^3 \Rightarrow v(t) = 2t - 3t^2 \text{ m/s.}$

$= t(2-3t)$

(b) At what time t is the particle at rest?

$v = 0 \iff t(2-3t) = 0$

$\iff t = 0, \frac{2}{3}$ sec.

(c) What is the total distance traveled by the particle from time $t = 0$ to time $t = 1$? (*Hint:* The correct answer is bigger than 0.)

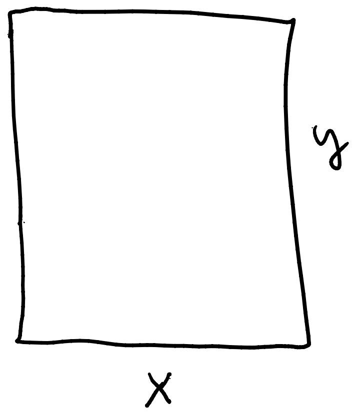
Dist on $0 \le t \le 1$ = Dist. on $0 \le t \le \frac{2}{3}$ + Dist on $\frac{2}{3} \le t \le 1$

$= \left|s(\tfrac{2}{3}) - s(0)\right| + \left|s(1) - s(\tfrac{2}{3})\right|$

$= \left|(\tfrac{2}{3})^2(1-\tfrac{2}{3}) - 0\right| + \left|0 - (\tfrac{2}{3})^2(1-\tfrac{2}{3})\right|$

$= \boxed{\frac{8}{27} \text{ m}}$

4. (12 points) The width of a rectangle grows at a rate of 1 inch per second, and the height of the rectangle is growing at a rate of 2 inches per second. How fast is the area of the rectangle growing when the width is 4 inches and the height is 5 inches?



Given: $\frac{dx}{dt} = 1$ inch/s

$\frac{dy}{dt} = 2$ inch/s

Goal: $\left.\frac{dA}{dt}\right|_{(x,y)=(4,5)}$.

$$A = xy \Rightarrow \frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt} = y + 2x$$

$$\Rightarrow \left.\frac{dA}{dt}\right|_{(x,y)=(4,5)} = 5 + 2(4) = 13 \text{ in}^2/\text{s}.$$

5. (12 points) If $f(x) = e^x/x$, find the equation of the tangent line to the curve $y = f(x)$ at the point $(2, e^2/2)$.

$$f'(x) = \frac{e^x x - e^x}{x^2} \Rightarrow f'(2) = \frac{2e^2 - e^2}{4} = \frac{e^2}{4}$$

$$\Rightarrow y = \frac{e^2}{2} + \frac{e^2}{4}(x-2) = \frac{e^2}{2} + \frac{e^2}{4}x - \frac{e^2}{2} = e^2/4\, x$$

$$\boxed{y = \frac{e^2}{4}x}$$

6. (12 points) Use the linearization of the function $f(x) = \sqrt{x}$ at the point $a = 16$ to approximate $\sqrt{16.12}$.

$\rightarrow \quad f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$

$f(16) = \sqrt{16} = 4.$

$\Rightarrow \quad L(x) = f(16) + f'(16)(x-16)$

$= 4 + \frac{1}{8}(x-16)$

Then, $\sqrt{16.12} = f(16.12) \approx L(16.12) = 4 + \frac{1}{8}(16.12-16)$

$= 4 + \frac{0.12}{8}$

$= 4.015$

7. (13 points) Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$. Find the absolute maximum and absolute minimum values of f on the interval $[-2, 3]$. Your answer must be fully justified and guesswork will not be rewarded.

$\rightarrow \quad f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$

$= 12x(x-2)(x+1)$

Critical pts: $x = -1, 0, 2.$

Since f is continuous we can compare values at crit. pts. and endpts. of closed interval:

$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 1 = 3 + 4 - 12 + 1 = -4$

$f(0) = 1$

$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 1 = 48 - 32 - 48 + 1 = -31$

Endpts: $f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 + 1 = 48 + 32 - 48 + 1 = 33$

$$f(3) = 3(3)^4 - 4(3)^3 - 12(3)^2 + 1 = 243 - 108 - 108 + 1 = 28$$

Abs. max $f(-2) = 33$

Abs. min $f(2) = -31$.