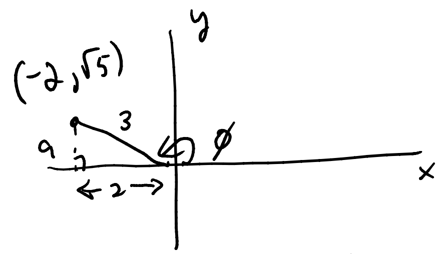
**1. (12 points)** Answer each part below and fully justify your answers. Put your final answer in the answer box:

(a) If $|\cos\phi| = 2/3$, and $\pi/2 < \phi < \pi$, then compute all six trig functions of $\phi$.




$a^2 + 2^2 = 3^2 \Rightarrow a = \sqrt{5}.$

Then,

$$\boxed{\sin\phi = \frac{\sqrt{5}}{3},\ \cos\phi = -2/3,\ \tan\phi = \frac{-\sqrt{5}}{2} \\ \csc\phi = \frac{3}{\sqrt{5}},\ \sec\phi = -3/2,\ \cot\phi = \frac{-2}{\sqrt{5}}}$$

(b) Find all $x$, $0 \le x \le 2\pi$, such that $\sin 2x = \cos x$.

$\sin(2x) = \cos x \Rightarrow 2\sin x\cos x = \cos x$

$\Rightarrow (2\sin x - 1)\cos x = 0$

$\Rightarrow \sin x = 1/2$ OR $\cos x = 0$

$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$ $\qquad \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

$$\boxed{x = \frac{\pi}{6}, 5\pi/6, \pi/2, 3\pi/2}$$

**2. (12 points)** Let $f(x) = \ln(e^x - 3)$. Answer each part below, show your work, and put the final answer in the answer box.

(a) Find an explicit formula for $f^{-1}(x)$.

$\rightarrow y = \ln(e^x - 3) \Rightarrow e^y = e^x - 3$

$\Rightarrow e^y + 3 = e^x$

$$\Rightarrow \quad \ln(e^y+3)=x$$

So, $\boxed{f^{-1}(x)=\ln(e^x+3)}$

(b) Find the domain and range of $f$ and $f^{-1}$ with brief justification.

$\rightarrow$ $x$ in $\mathrm{dom}(f) \iff e^x-3>0$

$\iff e^x>3$

$\iff x>\ln(3)$

On $x>\ln(3)$, range of $e^x-3$ is $(0,\infty)$, hence range of $f(x)=\ln(e^x-3)$ is $(-\infty,\infty)$.

So, $\mathrm{dom}(f)=(\ln(3),\infty)$

$\mathrm{range}(f)=(-\infty,\infty)$.

Then, $\mathrm{dom}(f^{-1})=\mathrm{range}(f)=(-\infty,\infty)$

$\mathrm{range}(f^{-1})=\mathrm{dom}(f)=(\ln(3),\infty)$

**3. (24 points)** Evaluate the following limits using limit laws and properties of limits to justify your answers. If they do not exist, explain why not. If the limit is $+\infty$ or $-\infty$, state which it is. Put your final answer in the answer box.

(a) $\lim\limits_{x\to 1^+} \dfrac{1-\sqrt{x}}{x-x^2}$. $\sim 0/0$

$$= \lim_{x\to 1^+} \frac{1-\sqrt{x}}{x(1-x)} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}}$$

$$= \lim_{x\to 1^+} \frac{1-x}{x(1-x)(1+\sqrt{x})}$$

$$= \lim_{x\to 1^+} \frac{1}{x(1+\sqrt{x})} = \boxed{\frac{1}{2}}$$

(b) $\lim\limits_{x\to 5^-} \dfrac{2}{(x-5)^3}$. $\sim \dfrac{2}{0} \to +\infty$ or $-\infty$

but which one?

For $x \to 5^-$, $x < 5 \Rightarrow (x-5)^3 < 0$

$$\Rightarrow \frac{2}{(x-5)^3} < 0.$$

Therefore, $\boxed{\lim\limits_{x\to 5^-} \dfrac{2}{(x-5)^3} = -\infty}$

(c) $\lim\limits_{x\to 0^-} \dfrac{|x|}{x^2} + \dfrac{1-2x^2}{x}$. $= \lim\limits_{x\to 0^-} \left[\dfrac{(-x)}{x^2} + \dfrac{1-2x^2}{x}\right]$

$$= \lim_{x\to 0^-} \left(-\frac{1}{x} + \frac{1}{x} - 2x\right)$$

$$= \lim_{x\to 0^-} (-2x)$$

$$= \lim_{x \to 0^-} (-2x)$$

$$= \boxed{0}$$

(d) $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)\cos(x)$

Since $\frac{1}{x} \to 0$ as $x \to \infty$,

$\sin\left(\frac{1}{x}\right) \to \sin(0) = 0$.

$-1 \leq \cos x \leq 1$

$\Rightarrow -\sin\left(\frac{1}{x}\right) \leq \sin\left(\frac{1}{x}\right)\cos(x) \leq \sin\left(\frac{1}{x}\right)$

for $x$ large, hence

$\boxed{\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)\cos x = 0}$ by

Squeeze th'm since

$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} -\sin\left(\frac{1}{x}\right) = 0.$

**4. (12 points)** Consider the following function:

$$f(x) = \begin{cases} -5-x & \text{if } x < -1 \\ A & \text{if } x = -1 \\ x^2+2x-3 & \text{if } -1 < x < 0 \\ B & \text{if } x = 0 \\ e^x + 1 & \text{if } x > 0 \end{cases}$$

(a) Compute each of the following or state why they do not exist.

$\lim_{x\to -1^-} f(x) = -5-(-1)$
$= -4$

$\lim_{x\to -1^+} f(x) = (-1)^2+2(-1)-3$
$= -4$

$\lim_{x\to -1} f(x) = -4$

$\lim_{x\to 0^-} f(x) = 0^2+2(0)-3$
$= -3$

$\lim_{x\to 0^+} f(x) = e^0+1$
$= 2$

$\lim_{x\to 0} f(x) =$ DNE

since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

(b) Find a value A such that $f(x)$ is continuous at $x = -1$ or explain why no such value exists. Similarly, find a value B such that $f(x)$ is continuous at $x = 0$ or explain why no such value exists. **Fully justify your answer.**

→ If $A=-4$, then $f(-1)=-4=\lim_{x\to -1} f(x)$, which means $f$ is continuous at $x=-1$.

There is no value of B that makes $f$ continuous at $x=0$ since $\lim_{x\to 0} f(x)$ does not exist.

**5. (16 points)** Consider the following function:

$$f(x) = \frac{(x+3)\sqrt{4x^2+1}}{x^2+x-6}.$$

(a) Determine whether or not the graph $y = f(x)$ has any vertical asymptotes. If so, state what they are. **Justify your answer using limits!**

$$f(x) = \frac{(x+3)\sqrt{4x^2+1}}{(x+3)(x-2)} = \frac{\sqrt{4x^2+1}}{x-2}$$

hence $\lim_{x \to -3} f(x)$ is finite (so $x=-3$ <u>not</u> a V.A.)

$$\lim_{x \to 2^+} \frac{\sqrt{4x^2+1}}{x-2} = +\infty$$

$$\lim_{x \to 2^-} \frac{\sqrt{4x^2+1}}{x-2} = -\infty$$ , hence

$$\boxed{f \text{ has a V.A. at } x=2.}$$

(b) Again referring to

$$f(x) = \frac{(x+3)\sqrt{4x^2+1}}{x^2+x-6},$$

determine whether or not the graph $y = f(x)$ has any horizontal asymptotes. If so, state what they are. **Justify your answer using limits!**

$$\lim_{x \to \infty} f = \lim_{x \to \infty} \frac{\sqrt{4x^2+1}}{x-2} \sim \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x^2}\sqrt{4+1/x^2}}{x(1-2/x)}$$

$$\lim_{x\to\infty} \quad x(1-2/x)$$

$$= \lim_{x\to\infty} \frac{|x|\sqrt{4+1/x^2}}{x(1-2/x)}$$

$x \to \infty,$
$\Rightarrow x > 0$
$\Rightarrow |x| = x \rightarrow$

$$= \lim_{x\to\infty} \frac{x\sqrt{4+1/x^2}}{x(1-2/x)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{4+\cancel{1/x^2}^{\to 0}}}{1-\cancel{2/x}^{\to 0}}$$

$$= 2$$

Similarly,

$$\lim_{x\to-\infty} f = \lim_{x\to-\infty} \frac{|x|\sqrt{4+1/x^2}}{x(1-2/x)}$$

$x \to -\infty,$
$\Rightarrow x < 0$
$\Rightarrow |x| = -x$

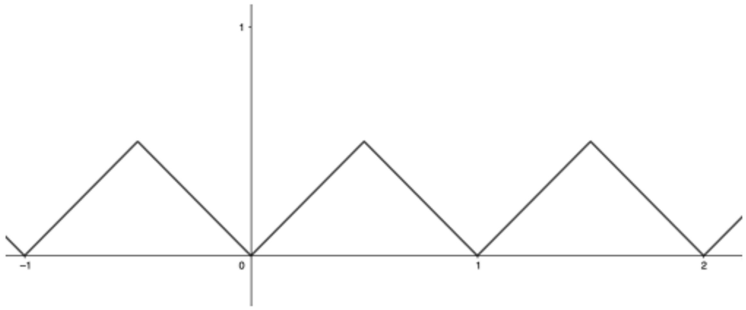
$$= \lim_{x\to-\infty} \frac{-x\sqrt{4+1/x^2}}{x(1-2/x)}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{4+\cancel{1/x^2}^{\to 0}}}{1-\cancel{2/x}^{\to 0}}$$

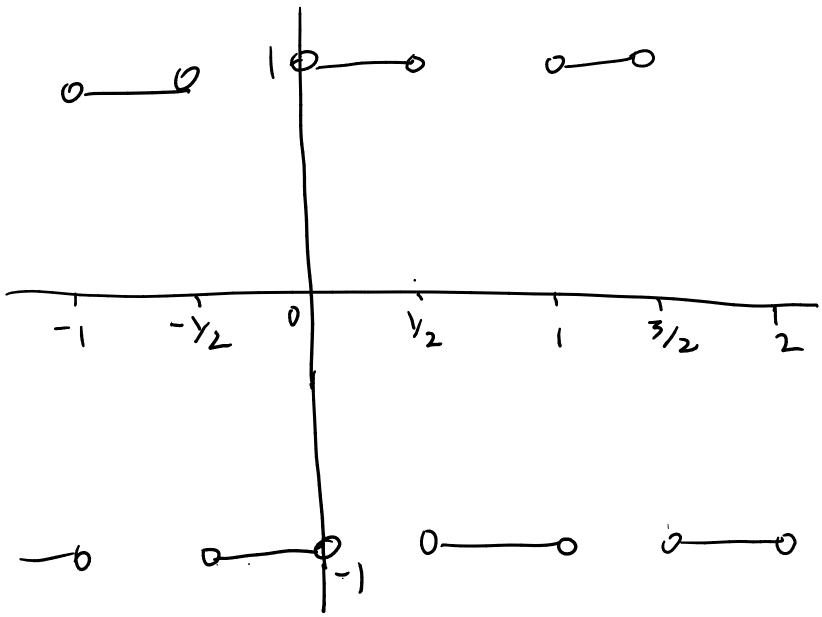
$$= -2$$

Therefore, $f$ has H. A.'s at $y=-2$ and $y=2$.

**6. (12 points)** Let $y = f(x)$ give the distance between $x$ and the nearest integer, graphed below.




(a) Graph the derivative $y = f'(x)$ on the axes below.




(b) Give all real values of $x$ in $(-\infty, \infty)$ at which $f$ is not differentiable. Only your answer will be considered, no justification is required.

$f'(x)$ does not exist at

$x = \ldots, -2, -3/2, -1, -1/2, 0, 1/2, 1, 3/2, \ldots$

**7. (12 points)** Answer each part below.

(a) Write the limit definition of the derivative $f'(a)$ of a function $f$ at $a$.

$$\rightarrow f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

OR

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(b) Now let $f(x) = x^2$. Use the limit definition of the derivative to find $f'(2)$.

$$\to \quad f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h}$$

$$= \lim_{h \to 0} \frac{h(4+h)}{h}$$

$$= \lim_{h \to 0} (4+h)$$

$$= 4$$

Note: no points awarded for simply using power rule.