**Part A**

**1. (8 points)** Find the limit or show that it does not exist. Justify your answer by using properties/theorems involving limits and/or continuous functions.

(a) $\lim_{x\to 2} \frac{\sqrt{x^2-4x+5}-1}{(x-2)^2}$

$\sim 0/0$

$$= \lim_{x\to 2} \frac{(\sqrt{x^2-4x+5}-1)(\sqrt{x^2-4x+5}+1)}{(x-2)^2(\sqrt{x^2-4x+5}+1)}$$

$$= \lim_{x\to 2} \frac{x^2-4x+5-1}{(x-2)^2(\sqrt{x^2-4x+5}+1)}$$

$$= \lim_{x\to 2} \frac{(x-2)^2}{(x-2)^2(\sqrt{x^2-4x+5}+1)}$$

$$= \lim_{x\to 2} \frac{1}{\sqrt{x^2-4x+5}+1} = \frac{1}{2}$$

(b) $\lim_{x\to\infty} \frac{\sin(\ln(x^2+e^x))}{x^2}$

$\to -1 \le \sin(\ln(x^2+e^x)) \le 1$ for all $x$

$\Rightarrow -\frac{1}{x^2} \le \frac{\sin(\ln(x^2+e^x))}{x^2} \le \frac{1}{x^2}$ for $x \neq 0$

Since $\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} -\frac{1}{x^2} = 0$,

we know $\lim_{x\to\infty} \frac{\sin(\ln(x^2+e^x))}{x^2} = 0$ by Squeeze Theorem.

**2. (10 points)** Let $f(x) = \frac{1}{1+2x}$ and recall the definition of the derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

(a) (6pts) Use the definition to find $f'(x)$. Note: No points will be awarded if you do not use the definition and you will lose significant points if you use L'Hospital's rule.

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(a) (6pts) Use the definition to find $f'(x)$. Note: No points will be awarded if you do not use the definition and you will lose significant points if you use L'Hospital's rule.

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{1+2(x+h)} - \frac{1}{1+2x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[ \frac{1+2x}{(1+2(x+h))(1+2x)} - \frac{1+2(x+h)}{(1+2(x+h))(1+2x)} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[ \frac{1+2x-1-2x-2h}{(1+2(x+h))(1+2x)} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{-2h}{(1+2(x+h))(1+2x)}$$

$$= \lim_{h \to 0} \frac{-2}{(1+2(x+h))(1+2x)} = \frac{-2}{(1+2x)^2}$$

(b) (4pts) Find the equation of the line tangent to the graph of $f(x)$ at $x = 0$.

$$f(0) = \frac{1}{1+2(0)} = 1$$

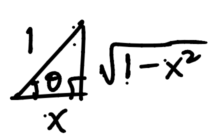
$$f'(0) = \frac{-2}{(1+2(0))^2} = -2$$

$$\Rightarrow \boxed{y = 1 - 2x}$$

**3. (8 points)** For this problem, justification is not required and partial credit will not be awarded. Complete each part below.

(a) Fully simplify $\tan(\arccos(x))$. Your answer must not contain any trigonometric functions to receive credit.

$\rightarrow$ let $\theta = \arccos x \Rightarrow \cos\theta = x$

$\Rightarrow$ 

$$\Rightarrow \tan(\arccos x) = \tan\theta$$
$$= \boxed{\frac{\sqrt{1-x^2}}{x}}$$

(b) Find $f'(x)$ where $f(x) = \sqrt{x}\ln(\sin x)$, where $0 < x < \pi$. You do **not** have to simplify your answer.

$$\boxed{f'(x) = \frac{1}{2}x^{-1/2}\ln(\sin x) + x^{1/2}\frac{1}{\sin x}\cos x}$$

(c) Find $f'(x)$ where $f(x) = \frac{x^2 e^x}{\sec x}$. You do **not** have to simplify your answer.

$$f'(x) = \frac{\frac{d}{dx}(x^2e^x)\sec x + x^2e^x\frac{d}{dx}(\sec x)}{\sec^2 x}$$
$$= \boxed{\frac{(2xe^x + x^2e^x)\sec x + x^2e^x\sec x\tan x}{\sec^2 x}}$$

(d) Find $f'(x)$ where $f(x) = x^{\frac{3}{2}}(1+\tan x)^{100}$. You do **not** have to simplify your answer.

$$\rightarrow \boxed{f'(x) = \frac{3}{2}x^{1/2}(1+\tan x)^{100} + x^{3/2}\cdot 100(1+\tan x)^{99}\sec^2 x}$$

**4. (8 points)** Find all pairs of numbers $(a, b)$ that make $f$ continuous

$$f(x) = \begin{cases} a^2x^2 - 3 & \text{if } x \leq 1 \\ -2x + b & \text{if } 1 < x \leq 2 \\ 2a - \sqrt{x+2} & \text{if } 2 < x \end{cases}$$

$$f(1) = \lim_{x\to 1^-} f(x) = a^2 - 3$$

$$\lim_{x\to 1^+} f(x) = -2 + b$$

$\rightarrow$ For continuity at $x=1$: $a^2 - 3 = b - 2$

$x \to 1^+$

$\Rightarrow$ For continuity at $x=1$: $a^2-3 = b-2$

$f(2) = \lim_{x\to 2^-} f(x) = -4+b$

$\lim_{x\to 2^+} f(x) = 2a-2$

$\Rightarrow$ For continuity at $x=2$: $2a-2 = b-4$

$2a-2=b-4 \Rightarrow 2a = b-2 = a^2-3$

$\Rightarrow a^2-2a-3=0 \Rightarrow (a-3)(a+1)=0$

$\Rightarrow a = -1, 3$

Then: $a=-1 \Rightarrow b = 2a+2$
$= 2(-1)+2$
$= 0$

$a=3 \Rightarrow b = 2a+2$
$= 2(3)+2$
$= 8$

Solutions: $(a,b) = (-1,0)$
and $(a,b) = (3,8)$.

**5. (8 points)** Find $dy/dx$ by implicit differentiation:

$$\cos(x^2+y^2) = xe^y.$$

$\Rightarrow \frac{d}{dx}\left[\cos(x^2+y^2)\right] = \frac{d}{dx}\left[xe^y\right]$

$$\Rightarrow -\sin(x^2+y^2)\frac{d}{dx}(x^2+y^2) = 1\cdot e^y + x e^y \frac{dy}{dx}$$

$$\Rightarrow -\sin(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = e^y + xe^y\frac{dy}{dx}$$

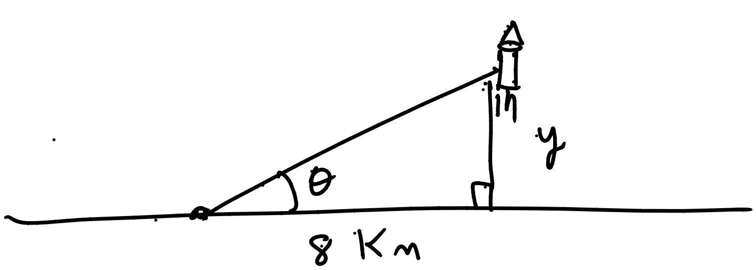
$$\Rightarrow -2x\sin(x^2+y^2) - 2y\sin(x^2+y^2)\frac{dy}{dx}$$
$$= e^y + x e^y \frac{dy}{dx}$$

$$\Rightarrow -2y\sin(x^2+y^2)\frac{dy}{dx} - xe^y\frac{dy}{dx}$$
$$= e^y + 2x\sin(x^2+y^2)$$

$$\Rightarrow \frac{dy}{dx}\left[-2y\sin(x^2+y^2) - xe^y\right]$$
$$= e^y + 2x\sin(x^2+y^2)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-\left(e^y + 2x\sin(x^2+y^2)\right)}{2y\sin(x^2+y^2) + xe^y}}$$

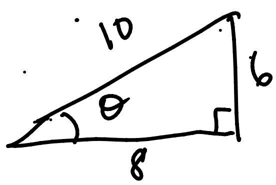
**6. (8 points)** A NASA enthusiast watches the latest rocket launch through a telescope from the ground 8 km from the launch pad. The enthusiast rotates her telescope to keep the rocket in sight at all times. How fast, in radians per second, is she rotating her telescope at the moment the rocket reaches an altitude of 6 km and has a speed of 1 km/s? **Fully simplify your answer.** (Assume that both the launch pad and the enthusiast are at sea level and that the rocket's trajectory is straight upwards.)



$$\tan\theta = \frac{y}{8} \Rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{1}{8}\frac{dy}{dt}$$

$$\Leftrightarrow \frac{d\theta}{dt} = \frac{\cos^2\theta}{8}\frac{dy}{dt}$$

When $y = 6\text{km}$:




$$\Rightarrow \cos\theta = \frac{8}{10} = \frac{4}{5}$$

$$\Rightarrow \left.\frac{d\theta}{dt}\right|_{\substack{y=6\text{km} \\ \frac{dy}{dt}=1\text{ km/s}}} = \frac{\left(\frac{4}{5}\right)^2}{8}\cdot(1) = \boxed{\frac{2}{25}\text{ Km/s}}$$

**Part B**

**1. (9 points)** Find the following limits or show they do not exist. If the limit is $+\infty$ or $-\infty$, determine which one. Justification is required, **show all work**.

(a) $\lim_{x\to\infty} x^3 e^{-x^2}$

$$= \lim_{x\to\infty} \frac{x^3}{e^{x^2}} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{3x^2}{2x\,e^{x^2}}$$

$$= \lim_{x\to\infty} \frac{3\cdot x}{2e^{x^2}}$$

$$\overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{3}{2(2x)e^{x^2}} = \boxed{0}$$

(b) $\lim_{x\to 0^+} \sqrt{x}\ln x$

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \overset{\infty/\infty}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}$$

$$= \lim_{x\to 0^+} \frac{\ }{x^{-1/2}} = \lim_{x\to 0^+} \frac{\ }{-\frac{1}{2}x^{-3/2}}$$

$$= \lim_{x\to 0^+} -\frac{1}{2}x^{1/2} = \boxed{0}$$

(c) $\lim_{x\to 0^+} (1+4x)^{3/x}$

$$= e^{\lim_{x\to 0^+} \ln\left[(1+4x)^{3/x}\right]}$$

$$= e^{\lim_{x\to 0^+} \frac{3\ln(1+4x)}{x}}$$

$$\overset{0/0}{=} e^{\lim_{x\to 0^+} \frac{3\frac{1}{1+4x}(4)}{1}}$$

$$= \boxed{e^{12}}$$

**2. (11 points)** Define the function

$$g(x) = \ln(9 - x^2).$$

(a) Find the domain of $g$.

$$9 - x^2 > 0 \Rightarrow x^2 < 9 \Rightarrow \boxed{-3 < x < 3}$$

(b) Find the vertical and horizontal asymptotes (if they exist).

Since domain is $(-3, 3) \rightarrow$ no horizontal asymptotes

Note: $\lim_{x\to 3^-} \ln(9-x^2) = -\infty$

and

$\lim_{x\to -3^+} \ln(9-x^2) = -\infty$

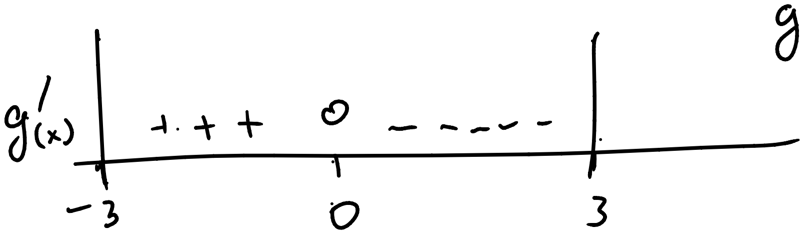
So, vertical asymptotes at $x = \pm 3$.

So, vertical asymptote at $x = -3$.

(c) Find the intervals of increase or decrease.

$$g'(x) = \frac{1}{9-x^2}(-2x) \Rightarrow g' = \text{DNE at } x = \pm 3$$
(not in domain)

$g' = 0$ at $x = 0$.




So, $g$ is increasing on $(-3, 0)$
and decreasing on $(0, 3)$

(d) Find the local maxima and local minima (if they exist).

$g$ has a local max at $x = 0$

$g(0) = \ln(9-0) = \ln(9)$

$\rightarrow$ local max at $(0, \ln(9))$.

(e) Find the intervals of concave upward and concave downward, and any inflection points (if they exist).

$$g'(x) = \frac{-2x}{9-x^2} \Rightarrow g''(x) = \frac{-2(9-x^2) - (-2x)(-2x)}{(9-x^2)^2}$$

$$= \frac{-18 + 2x^2 - 4x^2}{(9-x^2)^2}$$

$$= \frac{-2(x^2+9)}{(9-x^2)^2}$$

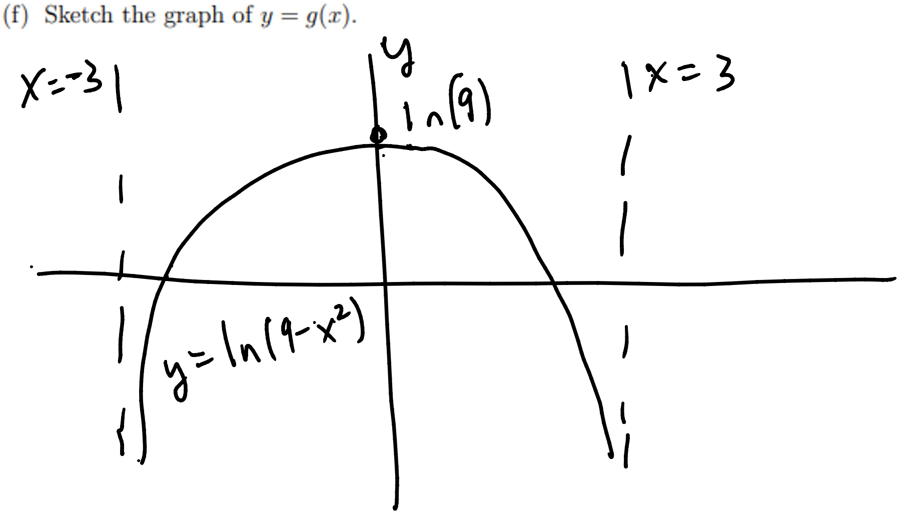
Then, $g''(x)$ exists for all $-3<x<3$

and $g''<0$ for all $-3<x<3$.

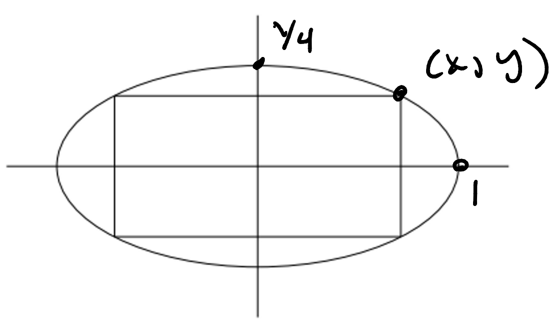
Then, $g$ is concave down on $(-3,3)$.

$g$ has no inflection pts.

(f) Sketch the graph of $y = g(x)$.




**3. (6 points)** Find the dimensions of the rectangle with the largest **perimeter** that can be inscribed in the ellipse $x^2 + 4y^2 = 1$.




$(x,y)$ satisfies $x^2+4y^2=1 \to y^2 = \frac{1-x^2}{4}$

$y = \frac{\sqrt{1-x^2}}{2}$ (since $y \geq 0$)

Perimeter $= 4x+4y$

$= 4x + 2\sqrt{1-x^2}$

Maximize $P(x) = 4x + 2\sqrt{1-x^2}$ on $0 \leq x \leq 1$.

$$P'(x) = 4 + 2\cdot\tfrac{1}{2}(1-x^2)^{-1/2}(-2x) = 4 - \frac{2x}{\sqrt{1-x^2}}$$

$P'$ exists on $0 < x < 1$.

$P' = 0$ when $2 = \dfrac{x}{\sqrt{1-x^2}} \Rightarrow 4 = \dfrac{x^2}{1-x^2}$

$$\Rightarrow 4 - 4x^2 = x^2$$

$$\Rightarrow 5x^2 = 4$$

$$\Rightarrow x = \frac{2}{\sqrt{5}}$$

Compare w/ endpts:

$$P\left(\frac{2}{\sqrt5}\right) = \frac{8}{\sqrt5} + 2\sqrt{1-\left(\frac{2}{\sqrt5}\right)^2} = \frac{8}{\sqrt5} + \frac{1}{\sqrt5} = \frac{9}{\sqrt5} = \sqrt{\frac{81}{5}}$$

$$P(0) = 2$$

$$P(1) = 4 + 2\sqrt{1-1^2} = 4$$

Since $\sqrt{\frac{81}{5}} > \sqrt{\frac{80}{5}} = 4 > 2$ we see max. occurs

when $x = \frac{2}{\sqrt5}$, $y = \dfrac{\sqrt{1-\left(\frac{2}{\sqrt5}\right)^2}}{2} = \dfrac{1}{2\sqrt5}$

$$\boxed{\text{width} = 2x = \frac{4}{\sqrt5} \qquad \text{height} = 2y = \frac{1}{\sqrt5}}$$

**4. (6 points)** Find the derivative of

$$g(x) = \int_{1-2x}^{1+2x} t^2 \sin(t^6)\, dt.$$

$$= \int^{0} t^2 \sin(t^6)\,dt + \int_0^{1+2x} t^2 \sin(t^6)\,dt$$

$$= \int_{1-2x} t^2 \sin(t^6)\,dt + \int_0 t^2 \sin(t^6)\,dt$$

$$= -\int_0^{1-2x} t^2 \sin(t^6)\,dt + \int_0^{1+2x} t^2 \sin(t^6)\,dt$$

let $y_1 = \int_0^{1-2x} t^2 \sin(t^6)\,dt = \int_0^{u} t^2 \sin(t^6)\,dt.$

and $u = 1-2x$

Then, $\dfrac{dy_1}{du} = u^2 \sin(u^6)$ and $\dfrac{du}{dx} = -2.$

$$\Rightarrow \frac{dy_1}{dx} = \frac{dy_1}{du}\frac{du}{dx} = -2u^2 \sin(u^6) = -2(1-2x)^2 \sin\left[(1-2x)^6\right]$$

let $y_2 = \int_0^{1+2x} t^2 \sin(t^6)\,dt = \int_0^{w} t^2 \sin(t^6)\,dt.$

and $w = 1+2x$

Then, $\dfrac{dy_2}{dw} = w^2 \sin(w^6)$ and $\dfrac{dw}{dx} = 2.$

$$\Rightarrow \frac{dy_2}{dx} = \frac{dy_2}{dw}\frac{dw}{dx} = 2w^2 \sin(w^6) = 2(1+2x)^2 \sin\left[(1+2x)^6\right]$$

Then, $g'(x) = -\dfrac{dy_1}{dx} + \dfrac{dy_2}{dx}$

$$\Rightarrow \boxed{g'(x) = 2(1-2x)^2 \sin\left[(1-2x)^6\right] + 2(1+2x)^2 \sin\left[(1+2x)^6\right]}$$

**5. (12 points)** Complete each part below.

(a) Evaluate $\int \frac{(1+r)^2}{\sqrt{r}}\, dr$.

$$= \int r^{-1/2}(1+2r+r^2)\,dr = \int \left(r^{-1/2} + 2r^{1/2} + r^{3/2}\right) dr$$

$$= \boxed{2r^{1/2} + \frac{4}{3}r^{3/2} + \frac{2}{5}r^{5/2} + C}$$

(b) Evaluate $\int xe^{-x^2}\, dx$.

let $u = -x^2 \Rightarrow du = -2x\,dx$

$-\frac{1}{2}\,du = x\,dx$

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C$$

$$= \boxed{-\frac{1}{2}e^{-x^2} + C}$$

(c) Evaluate $\int_{-1}^{1} e^{|x|}\, dx$.

$$= \int_{-1}^{0} e^{|x|}\,dx + \int_0^1 e^{|x|}\,dx$$

since $|x| = -x$ on $[-1, 0]$ and $|x| = x$ on $[0, 1]$

$$= \int_{-1}^{0} e^{-x}\,dx + \int_0^1 e^{x}\,dx$$

$$= -e^{-x}\Big|_{-1}^{0} + e^{x}\Big|_0^1$$

$$= -e^{-0} + e^{-(-1)} + e^{1} - e^{0}$$

$$= \boxed{2(e-1)}$$

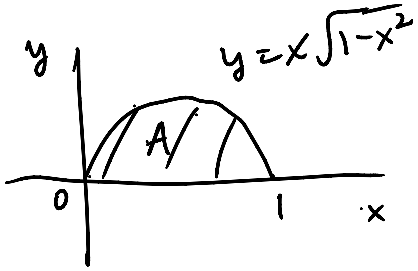
Sol'n #2: $e^{|x|}$ is an even fcn.

$$\Rightarrow \int_{-1}^{1} e^{|x|}\,dx = 2\int_0^1 e^{|x|}\,dx$$

$$= 2\int_0^1 e^{x}\,dx$$

$$= 2e^{x}\Big|_0^1 = 2(e-1)$$

(d) Find the area between the curve $y = x\sqrt{1-x^2}$ and the $x$-axis over the interval $[0, 1]$.




$$A = \int_0^1 x\sqrt{1-x^2}\,dx$$

$$= -\frac{1}{2}\int_{u=1}^{0} \sqrt{u}\,du$$

$$= \frac{1}{2}\int_0^1 u^{1/2}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^1$$

$$= \frac{1}{3}$$

let $u = 1 - x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$x = 0 \Rightarrow u = 1 - 0^2 = 1$

$x = 1 \Rightarrow u = 1 - 1^2 = 0$

**6. (6 points)** Let $f(x) = x^2 - x$.

(a) Find a formula for the right-hand Riemann sum with $n$ equal length sub-intervals, $R_n$, for $\int_0^2 f(x)\,dx$.
Hint: Your answer should be a formula in terms of $n$ and should not contain any summation symbols $\Sigma$. You may need some of the following formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $x_i = a + i\Delta x = \frac{2i}{n}$.

$\sum^{n} f\left(\frac{2i}{n}\right) 2$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2 - \frac{2i}{n}\right]$$

$$= \frac{2}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i\right]$$

$$= \frac{2}{n}\left[\frac{4}{n^2}\,\frac{n(n+1)(2n+1)}{6} - \frac{2}{n}\,\frac{n(n+1)}{2}\right]$$

$$= \frac{4}{3}\,\frac{(n+1)(2n+1)}{n^2} - \frac{2(n+1)}{n}$$

(b) Use your answer from part (a) to compute $\int_0^2 f(x)\,dx$. Answers using the Fundamental Theorem of Calculus will not be given credit.

$$\int_0^2 f(x)\,dx = \lim_{n\to\infty} R_n$$

$$= \frac{4}{3}\lim_{n\to\infty}\frac{2n^2+3n+1}{n^2} - 2\lim_{n\to\infty}\frac{n+1}{n}$$

$$= \frac{4}{3}\lim_{n\to\infty}\left(2+\cancel{\frac{3}{n}}^{\,0}+\cancel{\frac{1}{n^2}}^{\,0}\right) - 2\lim_{n\to\infty}\left(1+\cancel{\frac{1}{n}}^{\,0}\right)$$

$$= \frac{8}{3} - 2 = \frac{2}{3}$$