1. (17 points) Let $(y-1)(x^2+9)=10$.

(a) (5pts) Find $\frac{dy}{dx}$ by using implicit differentiation.

$$\frac{d}{dx}\left[(y-1)(x^2+9)\right] = \frac{d}{dx}(10)$$

$$\Rightarrow \frac{dy}{dx}(x^2+9) + (y-1)(2x) = 0$$

$$\Rightarrow \frac{dy}{dx}(x^2+9) = -2x(y-1)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-2x(y-1)}{x^2+9}}$$

(b) (5pts) Find the equations of all lines tangent to the curve $(y-1)(x^2+9)=10$ when $y=2$.

when $y=2 \rightarrow (2-1)(x^2+9)=10$

$\Rightarrow x^2=1$

$\Rightarrow x=\pm 1.$

At $(1,2)$: $\frac{dy}{dx} = \frac{-2(1)(2-1)}{1^2+9} = -\frac{1}{5}$

tangent line: $\boxed{y = 2 - \frac{1}{5}(x-1)}$

At $(-1,2)$: $\frac{dy}{dx} = \frac{-2(-1)(2-1)}{(-1)^2+9} = \frac{1}{5}$

tangent line: $\boxed{y = 2 + \frac{1}{5}(x+1)}$

(c) (5pts) Solve the same equation for y explicitly in terms of x and find $\frac{dy}{dx}$.

$$(y-1)(x^2+9) = 10 \Rightarrow y-1 = \frac{10}{x^2+9} \Rightarrow y = 1 + \frac{10}{x^2+9}$$

$$\boxed{\text{Then, } \frac{dy}{dx} = \frac{-10}{(x^2+9)^2}(2x) = \frac{-20x}{(x^2+9)^2}}$$

(d) (2pts) Show that implicit and explicit differentiation yield consistent results.

From part (a): $\frac{dy}{dx} = \frac{-2x(y-1)}{x^2+9} = \frac{-2x\left(\frac{10}{x^2+9}\right)}{x^2+9} = \frac{-20x}{(x^2+9)^2}$

Using $y = 1 + \frac{10}{x^2+9}$

which is consistent w/ part (c).

2. (16 points) Find the derivative of each function below.

(a) $f(x) = \arctan(3x^2 - 6)$

$$\Rightarrow f'(x) = \frac{1}{1+(3x^2-6)^2} \cdot (6x)$$

(b) $g(t) = \ln(5 + \sin^2 t)$

$$\Rightarrow g'(t) = \frac{1}{5+\sin^2 t} \cdot (2\sin t \cos t)$$

(c) $h(w) = \cos(e^{2w})$

$$\Rightarrow h'(w) = -\sin(e^{2w}) \cdot 2e^{2w}$$

(d) $k(z) = \frac{2z^2 - z^4 + z}{\sqrt{z}}$ $= \frac{2z^2}{z^{1/2}} - \frac{z^4}{z^{1/2}} + \frac{z}{z^{1/2}} = 2z^{3/2} - z^{7/2} + z^{1/2}$

$$\Rightarrow K'(z) = 3z^{1/2} - \frac{7}{2}z^{5/2} + \frac{1}{2}z^{-1/2}$$

3. (8 points) Find the derivative of $f(x)$ given below, where $c > 1$ is a positive constant.

$$f(x) = c^2 + c^c + 2^c + x^{\ln(cx)}$$

Note: since c is constant $\frac{d}{dx}(c^2 + c^c + 2^c) = 0$.

So $f'(x) = \frac{dy}{dx}$ where $y = x^{\ln(cx)}$.

$$\Rightarrow \ln y = \ln(x^{\ln(cx)}) = \ln(cx) \cdot \ln x$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{cx} \cdot c \cdot \ln x + \ln(cx) \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{\ln x + \ln(cx)}{x}\right]$$

$$\Rightarrow \boxed{\frac{dy}{dx} = x^{\ln(cx)} \cdot \left[\frac{\ln x + \ln(cx)}{x}\right]}$$

4. (16 points) Find the absolute extrema (absolute max and min points) of the following functions on the specified intervals, if there are any. If an absolute min or max does not exist, say so and briefly explain your reasoning. State the extrema in point notation, specifying x and y coordinates.

(a) $f(x) = e^x \cos x$ on $[0, 2\pi]$.

$\Rightarrow f'(x) = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$

$\underline{f' = 0}$: $e^x(\cos x - \sin x) = 0 \Rightarrow \cos x = \sin x$

$\Rightarrow x = \pi/4, 5\pi/4$

$f(\pi/4) = e^{\pi/4} \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} e^{\pi/4}$

$f(5\pi/4) = e^{5\pi/4} \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} e^{5\pi/4}$

$f(0) = e^0 \cos 0 = 1$

$f(2\pi) = e^{2\pi} \cos(2\pi) = e^{2\pi}$

$\boxed{\text{Abs. Min. at } \left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2} e^{5\pi/4}\right) \quad \text{Abs. Max at } (2\pi, e^{2\pi}).}$

(b) $g(x) = \sqrt{x} + |x|$ on $[0, 4)$.

On $[0,4)$, $|x| = x$.

So $g(x) = x^{1/2} + x$ on $[0,4)$.

Note that $g(x) \geq 0$ on $[0,4) \Rightarrow g(0) = \sqrt{0} + 0 = 0$ is abs. min. at $(0,0)$.

Also, g is strictly increasing on $(0,4)$ and g gets arbitrarily close to $g(4) = \sqrt{4} + 4 = 6$, but does not attain 6 on the interval $[0,4)$.

but does not attain 6 on the interval $[0,4)$.

So $g(x) < 6$ on $[0,4)$, hence g does not have an absolute max. on $[0,4)$.

5. (15 points) A particle travels along a line with respect to time (in seconds) according the following positions function:

$$s(t) = t^4 - 4t^3 + 4t^2 \quad (t \geq 0),$$

where $s(t)$ is in terms of meters.

(a) (6pts) When is the particle moving in the positive direction? When is the particle moving in the negative direction?

$v(t) = s'(t) = 4t^3 - 12t^2 + 8t$

$= 4t(t^2 - 3t + 2)$

$= 4t(t-2)(t-1) \Rightarrow v(t)$: $0 \; + \; 0 \; - \; 0 \; +$ at $0, 1, 2$

So, positive direction when $v > 0 \Rightarrow \boxed{(0,1) \cup (2,\infty)}$

negative direction when $v < 0 \Rightarrow \boxed{(1,2)}$.

(b) (6pts) Find the total distance the particle travels during the first 2 seconds?

Dist on $[0,2]$ = Dist on $[0,1]$ + Dist on $[1,2]$

$= |s(1) - s(0)| + |s(2) - s(1)|$

$= |1 - 0| + |0 - 1| =$ 2 meters

Note: $s(0) = 0$

$s(1) = 1^4 - 4 \cdot 1^3 + 4 \cdot 1^2 = 1$

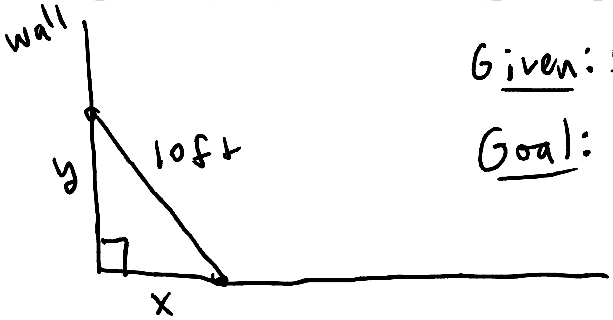
$s(2) = 2^4 - 4 \cdot 2^3 + 4 \cdot 2^2 = 0$.

(c) (3pts) Find the acceleration of the particle at $t = 2$ sec.

$a(t) = v'(t) = 12t^2 - 24t + 8$

$\Rightarrow a(2) = 12(2)^2 - 24(2) + 8 = 8 \text{ m/s}^2$

6. (14 points) A 10 ft long ladder rests against a vertical wall. Suppose the bottom of the ladder slides away from the wall at a rate of 2 ft/s. Consider the area of the triangle bounded by the wall, the ladder, and the ground. How fast is the area of this region changing when top of the ladder is 4 ft from the ground?



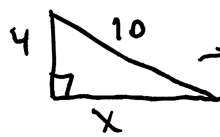
Given: $\frac{dx}{dt} = 2$ ft/s.

Goal: $\left.\frac{dA}{dt}\right|_{y=4\text{ ft}}$ where $A = xy$.

$$\Rightarrow \quad x^2 + y^2 = 100 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt}$$

Note: when $y = 4$ ft $\rightarrow$



$\rightarrow x = \sqrt{100-16} = \sqrt{84}$ ft.

Then $\left.\frac{dy}{dt}\right|_{y=4\text{ft}} = \frac{-\sqrt{84}}{4} \cdot (2) = \frac{-\sqrt{84}}{2}$ ft/s

Then, $A = \frac{xy}{2} \Rightarrow \frac{dA}{dt} = \left(\frac{dx}{dt}y + x\frac{dy}{dt}\right)\frac{1}{2}$

$$\Rightarrow \left.\frac{dA}{dt}\right|_{y=4\text{ft}} = \frac{2\cdot(4) + \sqrt{84}\left(\frac{-\sqrt{84}}{2}\right)}{2} = \frac{8 - \frac{84}{2}}{2} = -17 \text{ ft}^2/\text{s}$$

The area is decreasing at rate -17 ft^2/s when $y = 4$ ft.

7. (14 points) Let $f(x) = \dfrac{x+7}{3-x}$.

(a) (10pts) Use the limit definition of the derivative to find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+h+7}{3-(x+h)} - \frac{x+7}{3-x}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h+7)(3-x) - (x+7)(3-x-h)}{(3-x-h)(3-x)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{(x+7)(3-x)} + h(3-x) - \cancel{(x+7)(3-x)} + (x+7)h}{(3-x-h)(3-x)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{3h - hx + hx + 7h}{(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{10h}{(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{10}{(3-x-h)(3-x)}$$

$$= \frac{10}{(3-x)^2}$$

(b) (4pts) Find $f'(x)$ using derivative rules (i.e. quotient rule) and verify that your answer from part (a) is correct.

$$f(x) = \frac{x+7}{3-x} \Rightarrow f'(x) = \frac{1\cdot(3-x) - 1\cdot(x+7)\cdot(-1)}{(3-x)^2}$$

$$= \frac{3 - x + x + 7}{(3-x)^2}$$

$$= \frac{10}{(3-x)^2}$$ verified!