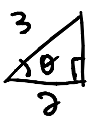
Midterm 1 Solutions

1. (12 points) Answer each part below and fully justify your answers:

(a) Find an integer A such that $A = \frac{1}{2}\log_3(16) - \log_3(12) - \frac{4}{3}\log_3(27)$.

$$\Rightarrow A = \log_3(16^{1/2}) - \log_3(12) - \frac{4}{3}(3)$$
$$= \log_3(4) - \log_3(12) - 4$$
$$= \log_3\left(\frac{4}{12}\right) - 4$$
$$= \log_3\left(\frac{1}{3}\right) - 4 = -1 - 4 = \boxed{-5}$$

(b) Given $\cos\theta = \frac{-2}{3}$, and $\sin\theta > 0$, compute $\sin(2\theta)$.

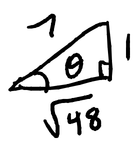


$\sqrt{9-4} = \sqrt{5} \rightarrow \sin\theta = \frac{\sqrt{5}}{3}$

$$\Rightarrow \sin(2\theta) = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{\sqrt{5}}{3}\right)\left(\frac{-2}{3}\right) = \frac{-4\sqrt{5}}{9}$$

(c) Find the exact value of $\tan(\sin^{-1}(1/7))$.

let $\theta = \sin^{-1}\left(\frac{1}{7}\right) \Rightarrow \sin\theta = \frac{1}{7}$



$$\Rightarrow \tan\left(\sin^{-1}\left(\frac{1}{7}\right)\right) = \tan\theta$$
$$= \frac{1}{\sqrt{48}}$$

2. (10 points) Complete each part below.

(a) Solve the following equation for x: $e^{2x} - 4e^x - 5 = 0$.

let $u = e^x \Rightarrow u^2 - 4u - 5 = 0$

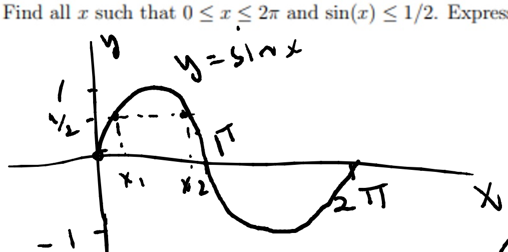
$(u-5)(u+1) = 0$

$u = -1, 5$

$e^x = -1 \rightarrow$ no x satisfies

$e^x = 5 \Rightarrow \boxed{x = \ln(5)}$

(b) Find all x such that $0 \leq x \leq 2\pi$ and $\sin(x) \leq 1/2$. Express your answer in terms of intervals.



$\sin x = 1/2$ on $[0, 2\pi]$

$\Rightarrow x = \pi/6, 5\pi/6$

So, $\sin x \leq 1/2$ on $[0, \pi/6] \cup [\frac{5\pi}{6}, 2\pi]$

3. (12 points) Let $f(x) = \frac{1}{1+e^x}$.

(a) What is the domain of f?

Note: $e^x > 0$ for all $x \Rightarrow e^x + 1 > 1$ for all x.

So, $1 + e^x \neq 0 \Rightarrow$ $\text{dom}(f) = (-\infty, \infty)$

(b) Find an explicit formula for the inverse of f.

$$x = \frac{1}{1+e^y} \Rightarrow 1 + e^y = \frac{1}{x}$$

$$\Rightarrow e^y = \frac{1}{x} - 1$$

$$\Rightarrow y = \ln\left(\frac{1}{x} - 1\right)$$

So, $f^{-1}(x) = \ln\left(\frac{1}{x} - 1\right)$

(c) What is the domain of $f^{-1}(x)$?

Need $\frac{1}{x} - 1 > 0 \Rightarrow \frac{1}{x} > 1$.

note: If $x \leq 0$, $\frac{1}{x} > 1$ is not satisfied.

If $x > 0$, $\frac{1}{x} > 1 \Rightarrow x < 1$.

$\text{dom}(f^{-1}) = (0, 1)$.

4. (12 points)

Given $f(x) = \sqrt{1-x}$, $g(x) = x^2 - 4$, find each of the following functions and their domains:

(a) $\frac{f}{g}$ $= \dfrac{\sqrt{1-x}}{x^2-4}$ $\rightarrow$ need $1-x \geq 0 \Rightarrow \underline{x \leq 1}$

and $x^2 - 4 \neq 0 \Rightarrow x \neq -2, 2.$

$\Rightarrow$ $\text{dom}\left(\frac{f}{g}\right) = (-\infty, -2) \cup (-2, 1]$

(b) $f \circ g$ $= f(g(x)) = f(x^2-4) = \sqrt{1-(x^2-4)} = \sqrt{5-x^2}$

$\rightarrow$ need $5 - x^2 \geq 0 \Rightarrow x^2 \leq 5$

$\Rightarrow |x| \leq \sqrt{5}$

$\Rightarrow -\sqrt{5} \leq x \leq \sqrt{5}.$

$\Rightarrow \text{dom}(f \circ g) = [-\sqrt{5}, \sqrt{5}].$

(c) $g \circ f$

$g(f(x)) = g(\sqrt{1-x}) = (\sqrt{1-x})^2 - 4$

$= 1 - x - 4$

$= -3 - x.$

$\rightarrow$ need x in $\text{dom}(f) \Rightarrow 1 - x \geq 0$

$\Rightarrow x \leq 1.$

So, $\text{dom}(g \circ f) = (-\infty, 1].$

5. (24 points) Evaluate the following limits if they exist. If they do not exist, explain why not. If the limit is $+\infty$ or $-\infty$, state which it is. Your answers must be fully justified to earn credit.

(a) $\lim_{x\to 3} \dfrac{2x^2-7x+3}{3-x}$ $= \lim_{x\to 3} \dfrac{(2x-1)(x-3)}{3-x}$

$$= \lim_{x\to 3} \frac{(2x-1)(x-3)}{-(x-3)}$$

$$= \lim_{x\to 3} -(2x-1) = \boxed{-5}$$

(b) $\lim_{x\to 1^+} \dfrac{|1-x|\,(x-1)}{\left|\frac{x-1}{|x-1|}\right| - x}$

$x\to 1^+ \Rightarrow x>1 \Rightarrow |1-x| = x-1$

Also, $\left|\dfrac{x-1}{|x-1|}\right| = \dfrac{|x-1|}{|x-1|} = 1.$

So, $\lim_{x\to 1^+} \dfrac{|1-x|(x-1)}{\left|\frac{x-1}{|x-1|}\right| - x} = \lim_{x\to 1^+} \dfrac{(x-1)^2}{1-x}$

$$= \lim_{x\to 1^+} -(x-1)$$

$$= 0$$

(c) $\lim_{x\to\infty} \ln(\sin(\arctan x))$. Hint: what is the limit of $\arctan x$?

$$\lim_{x\to\infty} \ln(\sin(\arctan x)) = \ln\left(\lim_{x\to\infty} \sin(\arctan x)\right)$$
$$= \ln\left(\sin\left(\lim_{x\to\infty} \arctan x\right)\right)$$
$$= \ln\left(\sin\left(\tfrac{\pi}{2}\right)\right)$$
$$= \ln(1)$$
$$= 0$$

(d) $\lim_{x\to\infty} \frac{\sin^4(3x)}{x^4+3x-1}$. Hint: try a squeeze argument.

$-1 \le \sin(3x) \le 1$ for all x.

$\Rightarrow$ $0 \le \sin^4(3x) \le 1$ for all x.

$\Rightarrow$ $0 \le \frac{\sin^4(3x)}{x^4+3x-1} \le \frac{1}{x^4+3x-1}$ for x large.

Then, since $\lim_{x\to\infty} 0 = 0$ and $\lim_{x\to\infty} \frac{1}{x^4+3x-1} = 0$,

we have $\lim_{x\to\infty} \frac{\sin^4(3x)}{x^4+3x-1} = 0$ by squeeze theorem.

6. (12 points) Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ \text{B} & \text{if } x = 0 \\ x^2 & \text{if } 0 < x < 2 \\ \text{C} & \text{if } x = 2 \\ x - 2 & \text{if } x > 2 \end{cases}$$

(a) Evaluate the following or state that they do not exist.

- $\lim_{x\to 0^-} f(x) = 0$
- $\lim_{x\to 0^+} f(x) = 0$
- $\lim_{x\to 0} f(x) = 0$
- $\lim_{x\to 2^-} f(x) = 4$
- $\lim_{x\to 2^+} f(x) = 0$
- $\lim_{x\to 2} f(x) =$ D.N.E.

(b) Find a value for B which makes f continuous at $x = 0$ or explain why no such value exists. Similarly, find a value for C which makes f continuous at $x = 2$ or explain why no such value exists. Fully justify your answer.

→ f is continuous at $x=0$ if $f(0) = \lim_{x\to 0} f(x)$. From above we know $\lim_{x\to 0} f(x) = 0$ and $f(0) = B$. Therefore $B = 0$ makes f continuous at $x = 0$.

→ Since $\lim_{x\to 2} f(x)$ does not exist, there is no value for $f(2) = C$ that makes $f(2) = \lim_{x\to 2} f(x)$.

So there is no C such that f is contin. at $x = 2$.

7. (18 points) Complete each part below.

(a) Let $f(x) = \sqrt{64x^2 + 4x} - 8x$, where $x \geq 1$. Determine whether or not f has a horizontal asymptote. If f has a horizontal asymptote, state what it is. **Justify your answer using limits.**

→ f is defined on $x \geq 1$ so there is no horiz. asymptote at $-\infty$.

At ∞: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\sqrt{64x^2+4x} - 8x\right)$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{64x^2+4x} - 8x\right)\cdot\left(\sqrt{64x^2+4x} + 8x\right)}{\sqrt{64x^2+4x} + 8x}$$

$$= \lim_{x\to\infty} \frac{64x^2+4x-64x^2}{\sqrt{64x^2+4x} + 8x}$$

$$= \lim_{x\to\infty} \frac{4x}{\sqrt{x^2\left(64+\frac{4}{x}\right)} + 8x}$$

$$= \lim_{x\to\infty} \frac{4x}{|x|\sqrt{64+4/x} + 8x}$$

$|x| = x$ for $x \to +\infty$

$$= \lim_{x\to\infty} \frac{4x}{x\sqrt{64+4/x} + 8x}$$

$$= \lim_{x\to\infty} \frac{4x}{x\left[\sqrt{64+4/x} + 8\right]}$$

$$= \lim_{x\to\infty} \frac{4}{\sqrt{64+\frac{4}{x}} + 8}$$

0

$$= \frac{4}{\sqrt{64}+8} = \frac{1}{4}$$

So f has a horizontal asymptote at $y = \frac{1}{4}$.

(b) Let $g(x) = \dfrac{x-2}{x^2-3x-2}$. Determine whether or not g has any vertical asymptotes. If g has any vertical asymptotes, state what they are. **Justify your answer using limits.**

Note: $x^2-3x-2=0 \Rightarrow x = \frac{1}{2}\left(3 \pm \sqrt{9+8}\right)$

$$= \frac{3 \pm \sqrt{17}}{2}.$$

So $$g(x) = \frac{x-2}{\left(x - \frac{3+\sqrt{17}}{2}\right)\left(x - \frac{3-\sqrt{17}}{2}\right)}$$

Since $\frac{3+\sqrt{17}}{2} = \frac{3}{2} + \frac{\sqrt{17}}{2} > \frac{3}{2} + \frac{1}{2} = 2$, we know $x-2>0$ near $x = \frac{3+\sqrt{17}}{2}$.

Also, $x-2<0$ near $x = \frac{3-\sqrt{17}}{2}$.

The signs of $g(x)$ are given by:

$g(x)$ $- - - - -$ DNE $+ + + +$ 0 $- - - - -$ DNE $+ + +$

$\frac{3-\sqrt{17}}{2}$ $\qquad$ 2 $\qquad$ $\frac{3+\sqrt{17}}{2}$

Then, $\displaystyle\lim_{x \to \left(\frac{3+\sqrt{17}}{2}\right)^+} g(x) = \infty$, $\displaystyle\lim_{x \to \left(\frac{3+\sqrt{17}}{2}\right)^-} g(x) = -\infty$

$\displaystyle\lim_{x \to \left(\frac{3-\sqrt{17}}{2}\right)^+} g(x) = \infty$, $\displaystyle\lim_{x \to \left(\frac{3-\sqrt{17}}{2}\right)^-} g(x) = -\infty$.

In particular, g has vertical asymptotes at $x = \frac{3+\sqrt{17}}{2}$ and $x = \frac{3-\sqrt{17}}{2}$.