Math 161: Calculus IA

Final Exam

December 16, 2010

NAME (please print legibly): Key

Your University ID Number: ______________________

Indicate your instructor with a check in the box:

Amanda Beeson	MWF 9:00 - 9:50 AM	
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Steve Lester	T, Th 2:00-3:15 PM	

- **The use of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the spaces provided.**
- **You are responsible for checking that this exam has all 19 pages.**

QUESTION	VALUE	SCORE
1	10	
2	16	
3	10	
4	8	
5	6	
TOTAL	50	

QUESTION	VALUE	SCORE
6	20	
7	10	
8	7	
9	6	
10	7	
TOTAL	50	

QUESTION	VALUE	SCORE
11	9	
12	9	
13	16	
14	16	
15	16	
16	20	
17	14	
TOTAL	100	

PART A.

1. (10 pts) Consider the function

$$g(x) = \begin{cases} x - 1 & \text{if } x < 4, \\ 5 - x^2 & \text{if } x \geq 4. \end{cases}$$

a) Evaluate $\lim_{x \to 4-} g(x)$.

$$= \lim_{x \to 4^-} x - 1 = 3$$

b) Evaluate $\lim_{x \to 4+} g(x)$.

$$= \lim_{x \to 4^+} 5 - x^2 = -11$$

c) Is this function continuous at $x = 3$? Justify your answer!

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} x - 1 = 2 = g(3)$$

thus the function is Continuous at 3

2. (16 pts) Differentiate the following functions.

a) $f(x) = 3x^2 + \sqrt{x}\cos(x)$

$f'(x) = 6x + \sqrt{x}\,(-\sin x) + \frac{1}{2}x^{-1/2}\cos x$

b) $f(x) = \dfrac{\ln(x)}{x}$

$f'(x) = \dfrac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \dfrac{1 - \ln x}{x^2}$

c) $f(x) = \arctan(e^{-x})$

$f'(x) = \dfrac{1}{1+(e^{-x})^2} \cdot e^{-x} \cdot (-1) = -\dfrac{e^{-x}}{1+e^{-2x}}$

d) Find $\dfrac{dy}{dx}$ where y is defined implicitly by $x^2 + xy + y^2 - 7 = 0$.

Take $\frac{d}{dx}$ both sides, we have

$2x + x\cdot\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$

$(x+2y)\frac{dy}{dx} = -y-2x$

$\dfrac{dy}{dx} = \dfrac{-y-2x}{x+2y}$

3. (10 pts) Let $f(x) = x^3 - 3x^2 - 24x + 2$.

a) Find an equation of the tangent line to the curve $y = f(x)$ at $x = 0$.

$f'(x) = 3x^2 - 6x - 24$.

$f'(0) = -24$. $\quad f(0) = 2$

point $(0, 2)$ $\quad$ slope -24

$y - 2 = -24x$

b) Find the point(s) at which $f(x)$ has a horizontal tangent line.

horizontal tangent line $\Leftrightarrow$ $f'(x) = 0$.

$3x^2 - 6x - 24 = 0$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

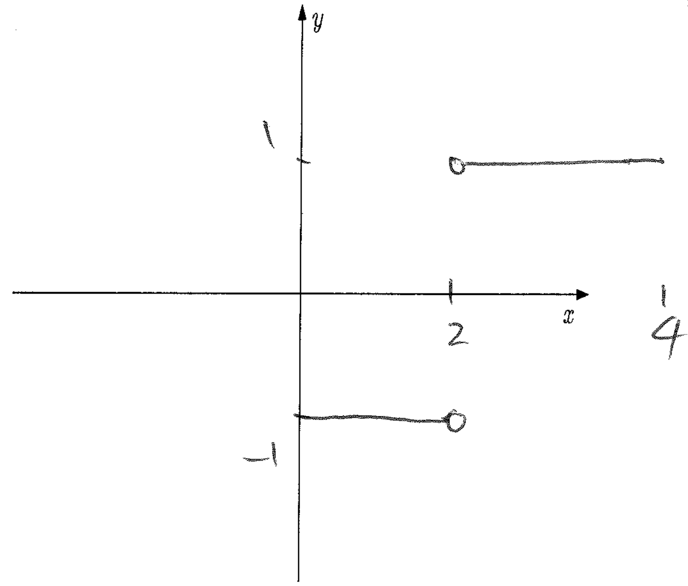
$\Rightarrow$ $x = 4$ or $x = -2$.

4. (8 pts) Suppose $f(x) = \dfrac{x-2}{|x-2|}$.

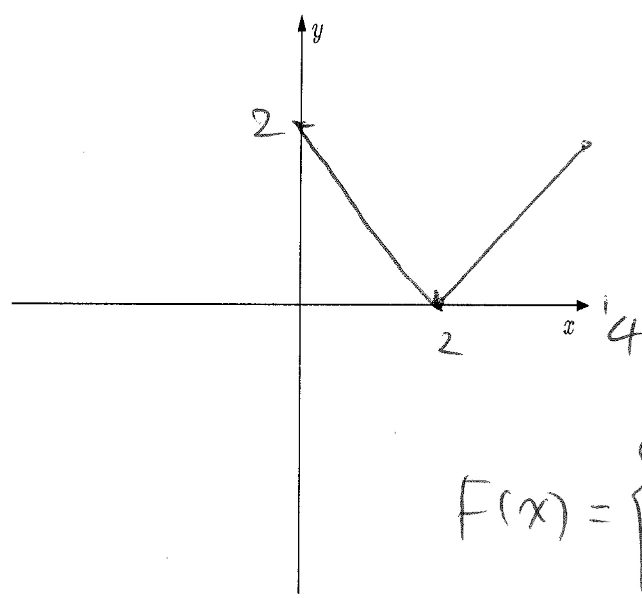
a) Sketch a graph of $f(x)$ on $[0, 4]$.

$$f(x) = \begin{cases} 1 & x > 2 \\ -1 & x < 2 \end{cases}$$

not defined at $x = 2$.



b) Now suppose $F(x)$ is a function with $F(2) = 0$ and $F'(x) = f(x)$ on $[0, 4]$. Sketch a graph of $F(x)$ on $[0, 4]$.



$$F(x) = \begin{cases} x - 2, & x \geq 2 \\ 2 - x, & x \leq 2 \end{cases}$$

5. (6 pts) Let $f(x) = \dfrac{6x^2+2}{(x^2-1)^3}$.

a) Find the horizontal asymptote(s), if any. Show your work. (You will not receive full credit if do not use limits to justify your answer.)

$$\lim_{x\to\infty} \frac{6x^2+2}{(x^2-1)^3} = \lim_{x\to\infty} \frac{12x}{3(x^2-1)^2 \cdot 2x} = \lim_{x\to\infty} \frac{2}{(x^2-1)^2}$$

$$= 0$$

l'Hopital's Rule

$$\lim_{x\to -\infty} \frac{6x^2+2}{(x^2-1)^3} = 0$$ similarly, thus $y=0$ is a horizontal asymptote.

b) Find the vertical asymptote(s), if any, and describe the behavior of $f(x)$ near the vertical asymptote(s). Show your work. (You will not receive full credit if do not use limits to justify your answer.)

$x^2-1=0$ gives rise to vertical asymptotes.

i.e., $x = \pm 1$

$$\lim_{x\to 1^+} \frac{6x^2+2}{(x^2-1)^3} = \infty \qquad \lim_{x\to 1^-} \frac{6x^2+2}{(x^2-1)^3} = -\infty$$

$$\lim_{x\to -1^+} \frac{6x^2+2}{(x^2-1)^3} = -\infty \qquad \lim_{x\to -1^-} \frac{6x^2+2}{(x^2-1)^3} = \infty$$

PART B.

6. (20 pts) Let $f(x) = 6x^5 - 10x^3$.

a) Find the x- and y-intercepts of the graph of $f(x)$.

X-intercepts: $0 = 6x^5 - 10x^3 = 2x^3(3x^2 - 5)$

$\Rightarrow x = 0, \pm\sqrt{\frac{5}{3}}$

y-intercepts: $y = 6 \cdot 0^5 - 10 \cdot 0^3 = 0$.

b) Is $f(x)$ odd? even? neither?

$f(-x) = 6(-x)^5 - 10(-x)^3 = -(6x^5 - 10x^3) = -f(x)$

thus odd function.

c) Find the absolute maximum and minimum of $f(x)$ on the interval $[-2, 0]$.

$f'(x) = 30x^4 - 30x^2$.

$30x^4 - 30x^2 = 0 \Rightarrow 30x^2(x^2 - 1) = 0$

$\boxed{x = 0, \pm 1}$ critical points.

absolute Min

only $-1, 0 \in [-2, 0]$

Compare: $f(-2) = 6 \cdot (-32) - 10(-8) = \boxed{-112}$

$f(-1) = 6(-1) - 10(-1) = \boxed{4}$

$f(0) = 0$.

absolut Max

d) Find the interval(s) on which $f(x)$ is increasing and the interval(s) on which it is decreasing.

from c)

	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
f'	+	−	−	+

Hence

f is increasing on $(-\infty,-1) \cup (1,\infty)$

and f is decreasing on $(-1,0)\cup(0,1)$

e) Find all critical numbers for $f(x)$ and classify any local maxima or minima.

from d) $0, \pm 1$ are critical points of f.

at $x=-1$, f has local maximum

at $x=0$, neither max nor min

at $x=1$, f has local minimum

f) Find the interval(s) on which $f(x)$ is concave up and the interval(s) on which it is concave down.

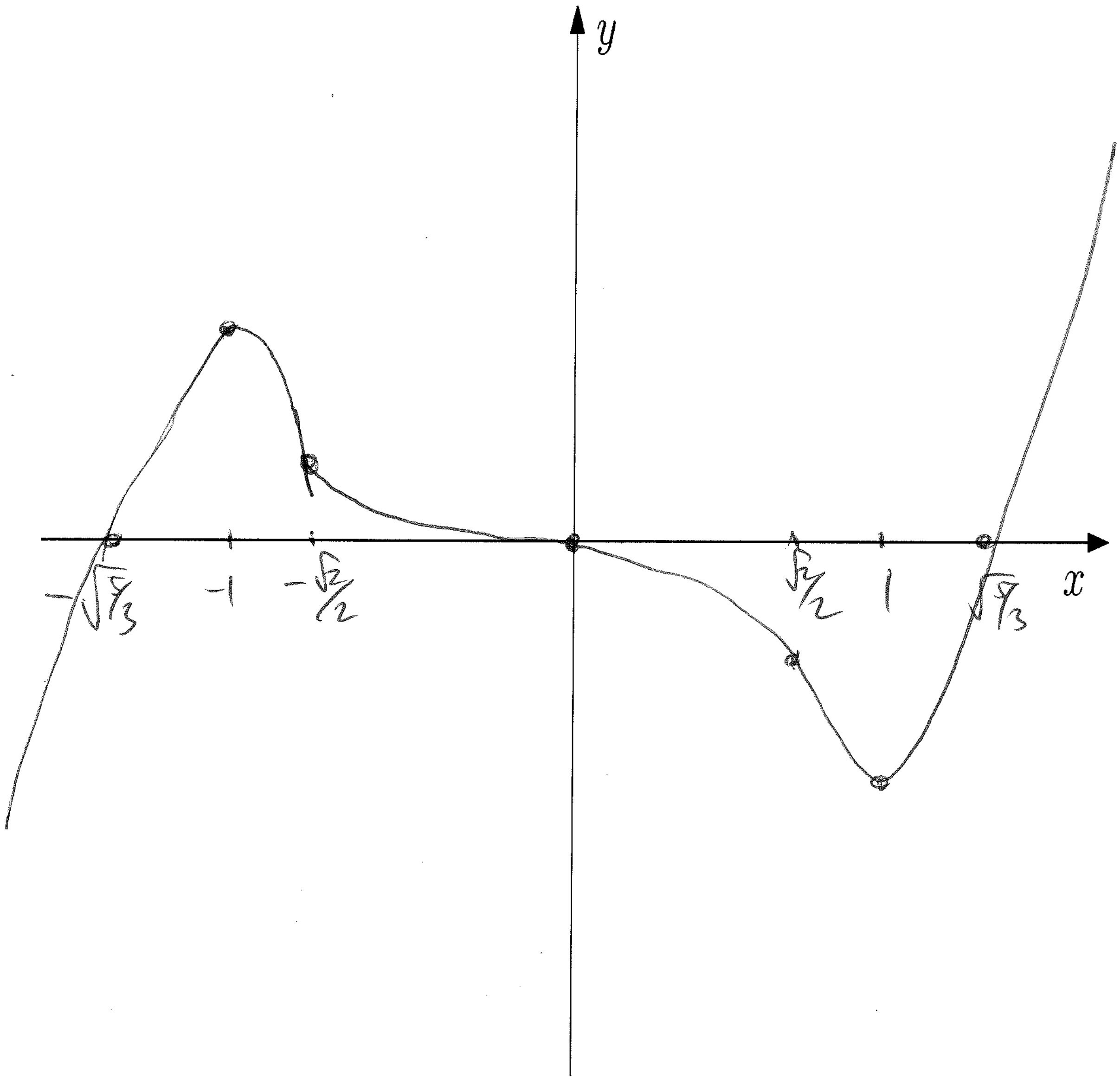
$f'' = 120x^3 - 60x$.

$f''=0 \Rightarrow 120x^3-60x=0$ $\quad 60x(2x^2-1)=0$

$\Rightarrow x=0, \pm\sqrt{\frac{1}{2}}$.

	$(-\infty,-\sqrt{\frac{1}{2}})$	$(-\sqrt{\frac{1}{2}},0)$	$(0,\sqrt{\frac{1}{2}})$	$(\sqrt{\frac{1}{2}},\infty)$
f''	−	+	−	+
	down	up	down	up

g) Use the information from the previous parts of the problem and the fact that the graph of $f(x)$ has inflection points at $(-\frac{\sqrt{2}}{2}, \frac{7\sqrt{2}}{4})$, $(0,0)$, and $(\frac{\sqrt{2}}{2}, -\frac{7\sqrt{2}}{4})$ to sketch the graph of $f(x)$. Label all inflection points, local maxima, and local minima.



7. (10 pts) Find the following limits. Write ∞ if it is infinity or DNE if it doesn't exist.

a) $\lim_{x\to 0} \frac{x^2+\sin(x)}{x}$ $= \lim_{x\to 0} x + \lim_{x\to 0} \frac{\sin x}{x} = 0+1 = 1$

b) $\lim_{x\to -\infty} \frac{x^2}{e^x+2}$

$= \cancel{\lim} \frac{\infty}{0+2} = \infty$

c) $\lim_{x\to 0+} x^x$

Consider $\lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} x \cdot \ln x$

$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$

L'Hospital's Rule

$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x$

$= 0$

So $\lim_{x\to 0^+} x^x = e^{\lim_{x\to 0} \ln x^x}$

$= e^0 = \boxed{1}$

8. (7 pts) Find the derivative of the function

$$y = \frac{(x^2+3)\cos(x)}{e^x \ln(x^2+3)}.$$

Logarithmic Differentiation:

$$\ln y = \ln(x^2+3) + \ln(\cos x) - \ln(e^x) - \ln(\ln(x^2+3))$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+3} - \frac{\sin x}{\cos x} - 1 - \frac{2x}{\ln(x^2+3)(x^2+3)}$$

$$\frac{dy}{dx} = \left(\frac{2x}{x^2+3} - \frac{\sin x}{\cos x} - 1 - \frac{2x}{(x^2+3)\ln(x^2+3)}\right) \cdot \frac{(x^2+3)\cos x}{e^x \ln(x^2+3)}$$

9. (6 pts) Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 25 cm^2?

$$S(t) = \ell(t)^2$$

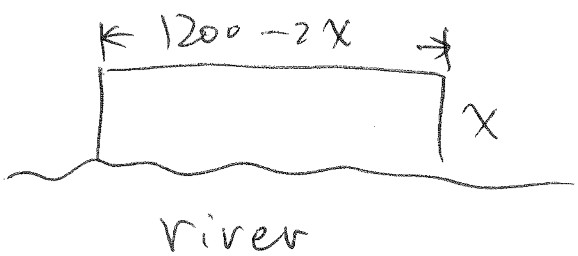
$$\frac{d}{dt} S(t) = 2\ell \frac{d\ell}{dt}$$

When $S(t) = 25 \text{ cm}^2$, $\ell(t) = 5$ cm.

So $\frac{d}{dt} S(t) = 2 \cdot 5 \cdot 6 = 60 \text{ cm}^2/\text{s}$

10. (7 pts) A farmer has 1200 ft of fencing and wants to fence off a rectangular field that borders a straight river. No fence is needed along the river. What are the dimensions of the field that has the largest area?

$X \in [0, 600]$



$$A(x) = x(1200 - 2x)$$

$$A'(x) = (1200 - 2x) + x(-2)$$

$$1200 - 2x - 2x = 0$$

$$\Rightarrow x = 300.$$

at $x = 300$

$A'' = -4 < 0$. so local maximum

at $x = 0$, and $x = 600$ extremal situation

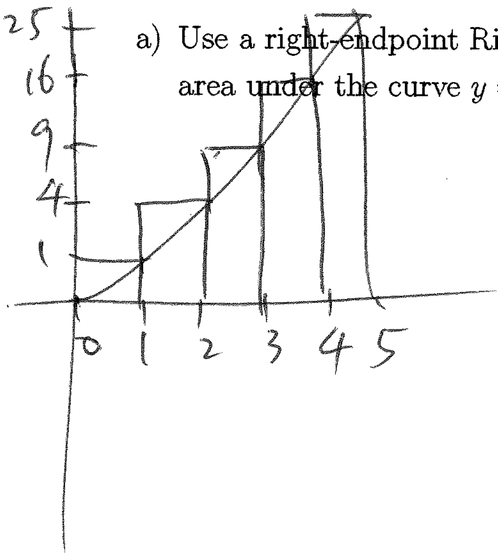
No area so

the dimension with largest area is 600×300

PART C.

11. (9 pts) Consider $f(x) = x^2$ on the interval $[0, 5]$.

a) Use a right-endpoint Riemann sum with 5 equal-length rectangles to approximate the area under the curve $y = x^2$ between $x = 0$ and $x = 5$.



$1 \cdot 1 + 1 \cdot 4 + 1 \cdot 9 + 1 \cdot 16 + 1 \cdot 25$

$= 55.$

b) Is your answer in (a) larger or smaller than $\int_0^5 x^2 dx$? Justify.

larger. Because x^2 is increasing.

Right end point given

overestimate.

12. (9 pts)

a) Express $\lim_{n\to\infty} \sum_{i=1}^{n} \left[(2\cancel{x_i} + 3) \left(\frac{4}{n} \right) \right]$ as a definite integral on the interval $[0, 4]$.

X_i should be $\frac{4i}{n}$.

Then $= \int_0^4 (2X+3)\,dx$

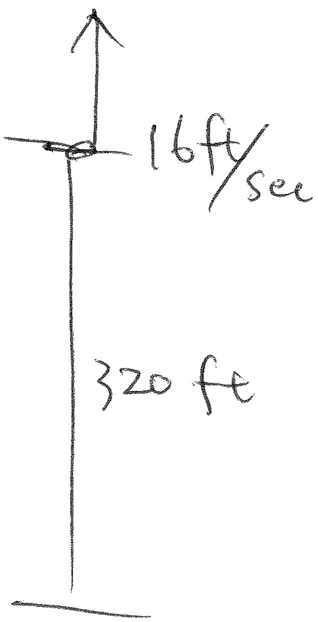
b) Evaluate the integral found in part (a).

$\int_0^4 (2x+3)\,dx = x^2 + 3x \Big|_0^4$

$= 28$

13. (16 pts) Suppose a ball is thrown upward from a 320 foot cliff with a velocity of 16 ft/sec.

a) Assuming that the air resistance can be ignored and that the acceleration due to gravity is -32 ft/sec^2, how high does it go? That is, find the maximum height.



$a = -32$

$V(t) = -32t + 16.$

$h(t) = -16t^2 + 16t + 320.$

highest point $\Leftrightarrow$ $V(t) = 0 \Rightarrow t = \frac{1}{2}.$

$h(\frac{1}{2}) = -4 + 8 + 320 = 324$ ft.

b) At what time does the ball hit the ground?

Time hits the ground $\Leftrightarrow$ $h(t) = 0$:

$-16t^2 + 16t + 320 = 0$

$\Rightarrow \quad -t^2 + t + 20 = 0$

$(5 - t)(t + 4) = 0 \quad \Rightarrow \quad t = 5$ sec

14. (16 pts)

a) If $F(x) = \int_2^x f(t)\,dt$, where $f(z) = \int_0^{z^2+1} \sin(u^2)\,du$, find the second derivative of F; that is, $F''(x)$.

$$F'(x) = f(x) = \int_0^{x^2+1} \sin(u^2)\,du$$

$$F''(x) = f'(x) = \cancel{\sin\left((z^2+1)^2\right)\cdot 2z}$$

$$= \sin\left((x^2+1)^2\right) \cdot 2x$$

b) Evaluate the integral

$$\int_0^4 \left[\frac{d}{dt}\left(\frac{\ln(t^3+1)}{t^2+1}\right)\right] dt.$$

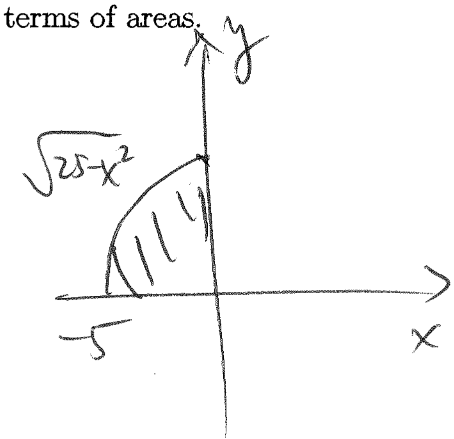
Fundamental Theorem of Calculus $\longrightarrow$

$$= \left.\frac{\ln(t^3+1)}{t^2+1}\right|_0^4$$

$$= \frac{\ln 65}{17} - \frac{\ln 1}{1} = \frac{\ln 65}{17}$$

15. (16 pts) Evaluate the following integrals by interpreting each in terms of areas.

a) $\displaystyle\int_{-5}^{0} \left(2 + \sqrt{25 - x^2}\right)\, dx$



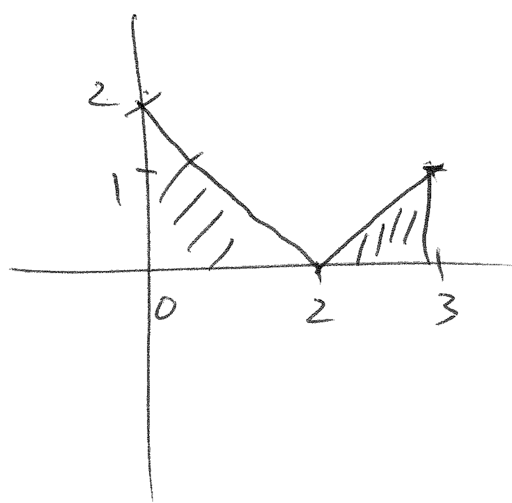
$$= \int_{-5}^{0} 2\, dx + \int_{-5}^{0} \sqrt{25 - x^2}\, dx$$

$$= 2 \cdot 5 + \frac{1}{4} \pi (5)^2$$

$$= 10 + \frac{25}{4}\pi$$

quarter disk.

b) $\displaystyle\int_{0}^{3} |x - 2|\, dx$



$$= 2 \cdot 2 \cdot \frac{1}{2} + 1 \cdot 1 \cdot \frac{1}{2}$$

$$= \frac{5}{2}$$

16. (20 pts) Evaluate the following integrals.

a) $\int_1^e \frac{1}{x}\,dx$ $= \ln|x| \Big|_1^e = 1$

b) $\int_0^\pi (\sin(x) + \cos(x))\,dx$

$= -\cos x + \sin x \Big|_0^\pi = (1+0) - (-1+0) = 2$

c) $\int \frac{x^2+8}{x^2}\,dx$ $= \int \frac{x^2}{x^2}\,dx + \int \frac{8}{x^2}\,dx$

$= x + 8\frac{x^{-1}}{-1} = x - 8x^{-1} + C$

d) $\int (e^x + \frac{2}{1+x^2})\,dx$

$= e^x + 2\tan^{-1}(x) + C$

17. (14 pts) A particle moves along a line so that its velocity a time t is $v(t) = t^4 - 4t$ (measured in meters per second).

a) Find the displacement of the particle during the time interval $1 \leq t \leq 3$.

$$\text{displacement} = \int_1^3 t^4 - 4t \, dt$$

$$= \frac{t^5}{5} - 4\frac{t^2}{2} \Big|_1^3$$

$$= \left(\frac{243}{5} - 2 \cdot 9\right) - \left(\frac{1}{5} - 2\right)$$

$$= \frac{242}{5} - 16$$

b) Find the distance traveled during the time period $1 \leq t \leq 3$.

$$\text{distance} = \int_1^3 |t^4 - 4t| \, dt.$$

$t^4 - 4t = 0$

$(t^3 - 4)t = 0$

$t = 0$ or $t = \sqrt[3]{4}$.

$$= \int_1^{\sqrt[3]{4}} \quad dt + \int_{\sqrt[3]{4}}^3$$

Not good numbers.

Change it to $v(t) = t^3 - 4t$.

So $t^3 - 4t = 0 \Rightarrow t = 0, \pm 2$.

$t^3 - 4t > 0$ if $t > 2$

$t^3 - 4t < 0$ if $t < 2$

So

$$\int_1^3 |t^3 - 4t| \, dt$$

$$= \int_1^2 4t - t^3 \, dt + \int_2^3 t^3 - 4t \, dt$$

$$= 2t^2 - \frac{t^4}{4} \Big|_1^2 + \frac{t^4}{4} - 2t^2 \Big|_2^3$$

$$= 9$$