MTH 161

Midterm 2

Tuesday, November 7, 2017

Last Name (Family Name): Solutions

First Name: __________

Student ID Number: __________

Circle your instructor and class time:

Hambrook (MW 10:25) Hambrook (MW 2:00)

Lorman (TuTh 9:40) Lubkin (MW 9:00) Xi (TuTh 3:25)

Please read the following instructions very carefully:

- You have **75 minutes** to complete this exam.
- Only pens/pencils are allowed. The presence of notes, calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden. You may refer to the **formula sheet** on the last page of this exam.
- Show your work and justify your answers. If you need extra space, use the back of the previous page and **clearly indicate** that you have done so. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given. **Clearly circle or label your final answers.**
- Sign the following academic honesty statement: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.*

Signature: __________

QUESTION	VALUE	SCORE
1	20	
2	20	
3	12	
4	16	
5	16	
6	16	
TOTAL	100	

1. (20 points) Compute the following limits. You may ***not*** use L'Hôpital's rule. (On this and all problems, don't forget to show your work!)

(a) $\lim_{t\to\infty} \frac{t-2t^2}{3t+5t^2}$ $= \lim_{t\to\infty} \frac{-2t^2+t}{5t^2+t} = \lim_{t\to\infty} \frac{\frac{1}{t^2}(-2t^2+t)}{\frac{1}{t^2}(5t^2+t)}$

$$= \lim_{t\to\infty} \frac{-2+\frac{1}{t}}{5+\frac{1}{t}} = \frac{-2+\lim_{t\to\infty}\left(\frac{1}{t}\right)}{5+\lim_{t\to\infty}\left(\frac{1}{t}\right)} = \frac{-2+0}{5+0} = \boxed{\frac{-2}{5}}$$

(b) $\lim_{x\to 0} \frac{\sin^2(x)}{x}$ $= \lim_{x\to 0} \frac{\sin(x)\cdot\sin(x)}{x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot x\right)$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0} x\right) = 1\cdot 1\cdot 0 = \boxed{0}.$$

(c) $\lim_{x\to\infty} \frac{x-7x^2}{4x+1}$ $= \lim_{x\to\infty} \frac{-7x^2+x}{4x+1} = \lim_{x\to\infty} \frac{\frac{1}{x}(-7x^2+x)}{\frac{1}{x}(4x+1)}$

$= \lim_{x\to\infty} \frac{-7x+1}{4+\frac{1}{x}}$. Numerator $\to -\infty$ as $x \to \infty$

Denominator $\to 4$

So $\lim_{x\to\infty} \frac{-7x+1}{4+\frac{1}{x}} = \boxed{-\infty}$

(d) $\lim_{x\to-\infty} \frac{\sqrt{5x^8+1}}{7+3x+x^4}$ $= \lim_{x\to-\infty} \frac{\sqrt{x^8\left(5+\frac{1}{x^8}\right)}}{x^4+3x+7}$

$$= \lim_{x\to-\infty} \frac{x^4\sqrt{5+\frac{1}{x^8}}}{x^4+3x+7} = \lim_{x\to-\infty} \frac{x^4\sqrt{5+\frac{1}{x^8}}}{x^4\left(1+\frac{3}{x^3}+\frac{7}{x^4}\right)}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{5+\frac{1}{x^8}}}{\left(1+\frac{3}{x^3}+\frac{7}{x^4}\right)} = \frac{\sqrt{5+\lim_{x\to-\infty}\frac{1}{x^8}}}{1+3\lim_{x\to-\infty}\frac{1}{x^3}+7\lim_{x\to-\infty}\frac{1}{x^4}} = \frac{\sqrt{5+0}}{1+3\cdot0+7\cdot0}$$

(e) $\lim_{x\to-\infty} \sqrt{x^2+x}+x$

$$= \boxed{\sqrt{5}}$$

$$= \lim_{x\to-\infty} \frac{\left(\sqrt{x^2+x}+x\right)\left(\sqrt{x^2+x}-x\right)}{\left(\sqrt{x^2+x}-x\right)}$$

$$= \lim_{x\to-\infty} \frac{(x^2+x)-x^2}{\left(\sqrt{x^2+x}-x\right)} = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+x}-x}$$

$$= \lim_{x\to-\infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x}\right)}-x} = \lim_{x\to-\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x}}-x}$$

$x\to-\infty$ means $x<0$ means $|x|=-x$

$$= \lim_{x\to-\infty} \frac{x}{(-x)\sqrt{1+\frac{1}{x}}-x} = \lim_{x\to-\infty} \frac{x}{(-x)\left(\sqrt{1+\frac{1}{x}}+1\right)}$$

$$= \lim_{x\to-\infty} \frac{-1}{\sqrt{1+\frac{1}{x}}+1} = \frac{-1}{\sqrt{1}+1} = \boxed{-\frac{1}{2}}$$

3

2. (20 points) Compute the following derivatives.

(a) $\frac{d}{dx}(3x^2 - 2\sqrt{x})$ $= \frac{d}{dx}(3x^2) - \frac{d}{dx}(2x^{1/2})$

$= 6x - x^{-1/2} = \boxed{6x - \frac{1}{\sqrt{x}}}$

(b) $\frac{d}{dx}\ln(e^x + \sin x)$ $= \frac{d}{dx}(e^x + \sin x) \cdot \frac{1}{e^x + \sin x}$

$= \boxed{\frac{e^x + \cos x}{e^x + \sin x}}$

(c) $\frac{d}{dx}\left(\frac{\sec x}{\tan x}\right)$ $= \frac{d}{dx}\left(\frac{1}{\cos x} \cdot \frac{1}{\frac{\sin x}{\cos x}}\right) = \frac{d}{dx}\left(\frac{1}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{\sin x}\right)$

$= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{d}{dx}(\csc x) = \boxed{-\csc(x)\cot(x)}$

(d) $\frac{d}{dx}\cos(\sin(2x^3+5x-3))$ $= \frac{d}{dx}\left[\sin(2x^3+5x-3)\right]\cdot\left(-\sin(\sin(2x^3+5x-3))\right)$

$$= \frac{d}{dx}(2x^3+5x-3)\ \cos(2x^3+5x-3)\left(-\sin(\sin(2x^3+5x-3))\right)$$

$$= \boxed{-(6x^2+5)\cos(2x^3+5x-3)\sin(\sin(2x^3+5x-3))}$$

(e) $\frac{d}{dx}(\sin(x)^{\cos(x^2)})$

Set $y = \sin(x)^{\cos(x^2)}$

$$\ln y = \ln\left(\sin(x)^{\cos(x^2)}\right) = \cos(x^2)\cdot\ln(\sin x)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left[\cos(x^2)\ln(\sin x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(\cos(x^2)\right)\ln(\sin x) + \cos(x^2)\frac{d}{dx}\left(\ln(\sin x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = -2x\sin(x^2)\ln(\sin x) + \frac{\cos(x^2)\cos x}{\sin x}$$

so $\frac{dy}{dx} = y\left[-2x\sin(x^2)\ln(\sin x) + \frac{\cos(x^2)\cos x}{\sin x}\right]$

$$= \boxed{\sin(x)^{\cos(x^2)}\left[-2x\sin(x^2)\ln(\sin x) + \frac{\cos(x^2)\cos x}{\sin x}\right]}$$

3. (12 points) Suppose a sample of 1000 mg of Cobalt-Thorium-G, a fictional radioactive substance, decays to 125 mg after 279 years. Find the half-life of Cobalt-Thorium-G.

(*Remember:* The mass of a radioactive substance satisfies the law of exponential decay, $m(t) = m_0 e^{kt}$. And the half-life of a substance is the time needed for half of the initial amount to decay.)

$$m(t) = 1000\, e^{kt}$$

$$m(279) = 125 \rightsquigarrow 125 = 1000\, e^{279k}$$

$$\frac{125}{1000} = \frac{1}{8} = e^{279k}$$

$$\ln\left(\frac{1}{8}\right) = 279k$$

$$k = \frac{1}{279}\ln\left(\frac{1}{8}\right) = \frac{-\ln(8)}{279} = \frac{-\ln(2^3)}{279} = \frac{-3\ln(2)}{279}$$

So $k = \dfrac{-3\ln(2)}{279} = \dfrac{-\ln(2)}{93}$.

$$m(t) = 1000\, e^{-\frac{\ln(2)}{93}t} = 1000\left(e^{\ln(2)}\right)^{-\frac{t}{93}}$$

$$= 1000 \cdot 2^{-t/93} = 1000 \cdot \left(\frac{1}{2}\right)^{t/93}.$$

Find h such that $m(h) = \frac{1000}{2} = 500$

Solve $1000 \cdot \left(\frac{1}{2}\right)^{h/93} = 500$

$$\left(\frac{1}{2}\right)^{h/93} = \frac{1}{2} \rightsquigarrow \frac{h}{93} = 1 \rightsquigarrow \boxed{h = 93 \text{ years}}$$

4. (16 points) A ball is thrown vertically upward with a velocity of 10 ft/s. Its height after t seconds is given by

$$s(t) = 10t - 2t^2$$

(a) Find the maximum height reached by the ball.

Height is at a max when $v(t) = s'(t) = 0$

$s'(t) = 10 - 4t \stackrel{\cdot}{=} 0 \rightsquigarrow 4t = 10 \rightsquigarrow t = \frac{5}{2}$

So the max height is $s\left(\frac{5}{2}\right) = 10 \cdot \frac{5}{2} - 2 \cdot \left(\frac{5}{2}\right)^2 = 25 - \frac{25}{2}$

$= \frac{25}{2} = \boxed{12.5 \text{ ft}}$

(b) How long does it take for the ball to fall back to the ground?

Find t such that $s(t) = 0$

$10t - 2t^2 = 0$

$2t(5-t) = 0 \rightsquigarrow t = 0, 5$

start time

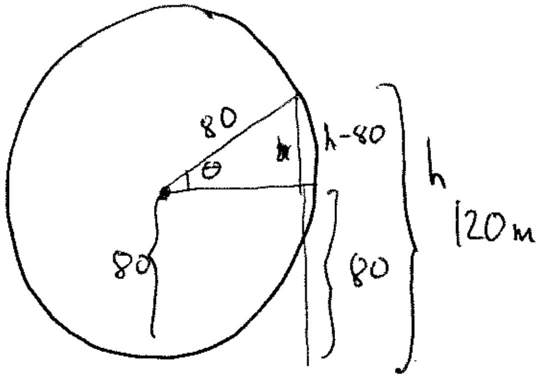
time when ball hits ground

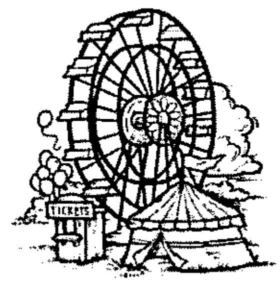
It takes $\boxed{5 \text{ seconds.}}$

(c) What is the ball's velocity at the moment that it hits the ground?

$s'(5) = 10 - 4 \cdot 5 = \boxed{-10 \text{ ft/s}}$

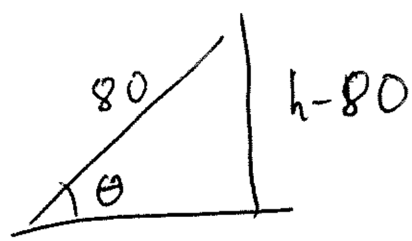
5. (16 points) A Ferris wheel with a radius of 80 m is rotating at a rate of one revolution every 3 minutes. How fast is a rider rising when her seat is 120 m above the ground?





Let θ = angle to horizontal
h = height of rider above the ground } functions of t.

Set up a right triangle



$$\sin\theta = \frac{h-80}{80}$$

$$\frac{d\theta}{dt}\cos(\theta) = \frac{1}{80}\frac{dh}{dt}$$

$$\frac{dh}{dt} = 80\,\frac{d\theta}{dt}\cos\theta$$

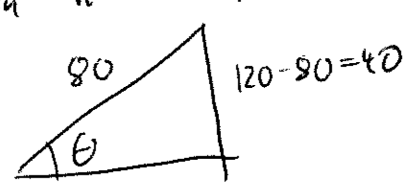
$$= 80\cdot\frac{2\pi}{3}\cdot\frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{80\pi\sqrt{3}}{3}\ \frac{\text{m}}{\text{min}}}$$

$$\frac{d\theta}{dt} = \frac{\text{one revolution}}{3\text{ min}}$$

$$= \frac{2\pi\text{ rad}}{3\text{ min}}$$

When $h = 120$,



$$\sin\theta = \frac{40}{80} = \frac{1}{2}$$

so $\theta = \frac{\pi}{6}$ and $\cos\theta = \frac{\sqrt{3}}{2}$

6. (16 points) Consider the curve defined by the equation

$$y^2 - 2y - 1 = 2x^3 + 3x^2$$

Find all points (x_0, y_0) at which the tangent line is horizontal. (*Hint: There will be four points. Start by using implicit differentiation to find* $\frac{dy}{dx}$ *as a function of* x *and* y.)

$$\frac{d}{dx}\left(y^2-2y-1\right) = \frac{d}{dx}\left(2x^3+3x^2\right)$$

$$2y\frac{dy}{dx} - 2\frac{dy}{dx} = 6x^2 + 6x$$

$$\frac{dy}{dx}(2y-2) = 6x^2+6x \rightarrow \frac{dy}{dx} = \frac{6x^2+6x}{2y-2}$$ factor out 2

$$\frac{dy}{dx} = \frac{3x^2+3x}{y-1}$$

~~Set~~ Tangent horizontal, so set $\frac{dy}{dx} = 0$

$$\frac{3x^2+3x}{y-1} = 0 \rightarrow 3x^2+3x = 0$$

$$3x(x+1)=0$$

So $x = 0, -1$. Need y coordinates

When $x=0$

$$y^2-2y-1=0$$

$$y = \frac{2 \pm \sqrt{4-4(-1)}}{2} = \frac{2\pm\sqrt{8}}{2} = \boxed{1\pm\sqrt{2}}$$

When $x=-1$

$$y^2-2y-1 = 2(-1)^3+3(-1)^2 = 1$$

$$y^2-2y-2=0$$

$$y = \frac{2\pm\sqrt{4-4(-2)}}{2} = 1\pm\sqrt{3}$$

Solutions: $(0, 1+\sqrt{2}), (0, 1-\sqrt{2}), (-1, 1+\sqrt{3}), (-1, 1-\sqrt{3})$

Scratch paper. You can tear this off if you want.