

MATH 161

Midterm 2

November 12, 2015

Please print your name and student ID again, and circle your instructor's name:

Name (please print): Solutions

University student ID: _____

Bobkova (TR 9:40) Bridy (MW 2:00) Doyle (MWF 10:25)

Hambrook (TR 3:25) Lubkin (MWF 9:00) Murphy (TR 4:50)

Please read the following instructions:

Only pens/pencils and a single 3 in. \times 5 in. index card with formulas are allowed. The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.

Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given. Clearly circle or label your final answers.

QUESTION	VALUE	SCORE
1	24	
2	18	
3	18	
4	20	
5	20	
TOTAL	100	

1. (24 points) For each of the following functions y , compute the derivative $\frac{dy}{dx}$.

$$(a) y = 2x^9 - 4\sqrt[5]{x^2} - \frac{4}{x^7} = 2x^9 - 4x^{2/5} - 4x^{-7}$$

$$\frac{dy}{dx} = \boxed{18x^8 - \frac{8}{5}x^{-3/5} + 28x^{-8}}$$

$$(b) y = (\ln(x))^x \rightsquigarrow \ln y = \ln((\ln(x))^x) \\ = x \ln(\ln(x))$$

$$\frac{1}{y} \frac{dy}{dx} = 1 \cdot \ln(\ln(x)) + x \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y \left(\ln(\ln(x)) + \frac{1}{\ln(x)} \right)$$

$$= \boxed{(\ln(x))^x \cdot \left(\ln(\ln(x)) + \frac{1}{\ln(x)} \right)}$$

$$(c) y = \cos(xe^x)$$

$$\frac{dy}{dx} = -\sin(xe^x) \cdot [1 \cdot e^x + x \cdot e^x]$$

$$= \boxed{-\sin(xe^x) \cdot (e^x + xe^x)}$$

$$(d) y = e^{\sin(\sqrt{x})}$$

$$\frac{dy}{dx} = \boxed{e^{\sin(\sqrt{x})} \cdot \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}}$$

$$(e) y = \frac{2 \tan(x)}{\ln(x) + e^x}$$

$$\frac{dy}{dx} = \boxed{\frac{(\ln(x) + e^x) \cdot 2 \sec^2(x) - 2 \tan(x) \cdot \left(\frac{1}{x} + e^x\right)}{(\ln(x) + e^x)^2}}$$

$$(f) y = \ln\left(\frac{(3x-3)^4 e^{2x}}{\sqrt[3]{2x+1}}\right) = \ln((3x-3)^4) + \ln(e^{2x}) - \ln((2x+1)^{1/3})$$
$$= 4 \ln(3x-3) + 2x - \frac{1}{3} \ln(2x+1)$$

$$\frac{dy}{dx} = 4 \cdot \frac{1}{3x-3} \cdot 3 + 2 - \frac{1}{3} \cdot \frac{1}{2x+1} \cdot 2$$

$$= \boxed{\frac{4}{x-1} + 2 - \frac{2}{3(2x+1)}}$$

2. (18 points)

(a) Suppose that $2y^2 + \sin(x) = xy - 8$. Find $\frac{dy}{dx}$.

$$\frac{d}{dx} (2y^2 + \sin(x)) = \frac{d}{dx} (xy - 8)$$

$$4y \frac{dy}{dx} + \cos(x) = y + x \frac{dy}{dx}$$

$$4y \frac{dy}{dx} - x \frac{dy}{dx} = y - \cos(x)$$

$$(4y - x) \frac{dy}{dx} = y - \cos(x)$$

$$\frac{dy}{dx} = \boxed{\frac{y - \cos(x)}{4y - x}}$$

(b) Consider the curve defined by the equation $x^3 + y^3 = 16$. Find all points where the tangent line to the curve is parallel to the line $x + y = 0$.

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(16)$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2}$$

$$\frac{dy}{dx} = -\frac{x^2}{y^2}$$

The line $x+y=0$ has slope -1 , so we want to know at which points on $x^3+y^3=16$ the tangent line has slope -1 :

$$\frac{dy}{dx} = -1 \implies -\frac{x^2}{y^2} = -1$$

$$\implies x^2 = y^2$$

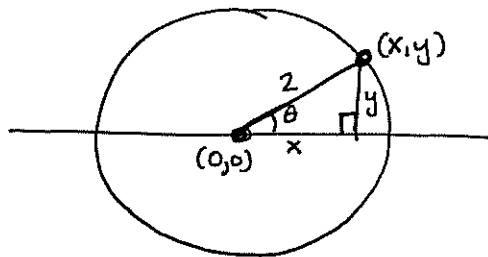
$$\implies x = \pm y$$

⊗ If $x=y$, then $x^3+y^3=16 \implies 2x^3=16 \implies x^3=8$
 $\implies x=2, y=2$

⊗ If $x=-y$, then $x^3+y^3=16 \implies x^3-x^3=16$
 $\implies 0=16 \leftarrow \text{cannot happen!}$

Therefore the only point whose tangent line has slope 1 is $\boxed{(2,2)}$.

3. (18 points) A 2 ft long rod has one end at the origin. The other end is the point (x, y) , which rotates counterclockwise in a circle around the origin. Assume that x is decreasing at a rate of 1 foot per minute when the angle between the rod and the positive x -axis is $\pi/6$. At that moment:



(a) What is the rate of change of y in feet per minute?

WANT : $\frac{dy}{dt} \Big|_{\theta = \pi/6}$

KNOW : $\frac{dx}{dt} \Big|_{\theta = \pi/6} = -1$;

$$x^2 + y^2 = 4$$

When $\theta = \pi/6$, $x = 2 \cos(\pi/6) = \sqrt{3}$
 $y = 2 \sin(\pi/6) = 1$.

$$x^2 + y^2 = 4$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$2(\sqrt{3})(-1) + 2(1) \frac{dy}{dt} = 0$$

$$2 \frac{dy}{dt} = 2\sqrt{3}$$

$$\frac{dy}{dt} = \boxed{\sqrt{3} \text{ ft/min}}$$

(b) How quickly is the rod rotating in radians per minute?

WANT : $\frac{d\theta}{dt} \Big|_{\theta = \pi/6}$

KNOW : $\frac{dx}{dt} \Big|_{\theta = \pi/6} = -1$

$$x = 2 \cos \theta$$

When $\theta = \pi/6$, $\sin(\theta) = \sin(\pi/6) = \frac{1}{2}$.

$$x = 2 \cos \theta$$

$$\frac{dx}{dt} = -2 \sin \theta \cdot \frac{d\theta}{dt}$$

$$-1 = -2 \left(\frac{1}{2}\right) \cdot \frac{d\theta}{dt}$$

$$1 = \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \boxed{1 \text{ radian/min}}$$

4. (20 points) A ball is thrown vertically from the window of a tall building. Its height above the ground (in feet) at time t seconds is given by the function

$$x(t) = -16t^2 + 32t + 48 : t \geq 0.$$

(a) On what interval(s) is the ball moving up?

The ball is moving up when velocity is positive:

$$v(t) = x'(t) = -32t + 32.$$

$$v(t) > 0 \iff -32t + 32 > 0$$

$$\iff 32 > 32t$$

$$\iff 1 > t.$$

Moving up : $[0, 1)$.

(b) When does the ball hit the ground?

$$x(t) = 0 \rightsquigarrow -16t^2 + 32t + 48 = 0$$

$$-16(t^2 - 2t - 3) = 0$$

$$-16(t-3)(t+1) = 0$$

$$t = -1, 3 \quad \leftarrow \text{only } 3 \text{ makes sense here}$$

$$t = \boxed{3 \text{ s}}$$

(c) What is the acceleration of the ball as a function of time?

$$a(t) = v'(t) = \boxed{-32 \text{ ft/s}^2}$$

(d) What is the total distance the ball travels before it hits the ground?

Moving up : $[0, 1)$

Moving down : $(1, 3)$

$$\begin{aligned} \text{distance upward} : x(1) - x(0) &= [-16(1)^2 + 32(1) + 48] - [-16(0)^2 + 32(0) + 48] \\ &= 64 - 48 \\ &= 16 \text{ ft} \end{aligned}$$

$$\begin{aligned} \text{distance downward} : x(1) - x(3) &= [-16(1)^2 + 32(1) + 48] - [-16(3)^2 + 32(3) + 48] \\ &= 64 - 0 \\ &= 64 \text{ ft.} \end{aligned}$$

$$\text{TOTAL: } 16 + 64 = \boxed{80 \text{ ft}}$$

5. (20 points)

- (a) A sample of radioactive material is placed in a room and begins decaying at a rate proportional to its mass. After 10 days, its mass equals $\frac{1}{5}$ of its starting mass. How many days does it take for the mass to equal $\frac{1}{10}$ of its starting mass?

Let $P(t)$ be the mass after t days. } Then $P(t) = P_0 e^{kt}$
Let P_0 be the initial mass. } for some constant k .

$$\bullet P(10) = \frac{1}{5} P_0 \implies P_0 e^{k \cdot 10} = \frac{1}{5} P_0$$

$$\implies e^{10k} = \frac{1}{5}$$

$$\implies 10k = \ln\left(\frac{1}{5}\right)$$

$$\implies k = \frac{1}{10} \ln\left(\frac{1}{5}\right).$$

$$\bullet P(t) = \frac{1}{10} P_0 \implies P_0 e^{\frac{1}{10} \ln\left(\frac{1}{5}\right) \cdot t} = \frac{1}{10} P_0$$

$$\implies e^{\frac{1}{10} \ln\left(\frac{1}{5}\right) \cdot t} = \frac{1}{10}$$

$$\implies \frac{1}{10} \ln\left(\frac{1}{5}\right) \cdot t = \ln\left(\frac{1}{10}\right)$$

$$\implies t = \boxed{\frac{10 \cdot \ln\left(\frac{1}{10}\right)}{\ln\left(\frac{1}{5}\right)} \text{ days}}$$

(b) Use the linearization of $f(x) = \sqrt[3]{x}$ at $x = 8$ to approximate $\sqrt[3]{8.2}$.

Linearization : $L(x) = f(8) + f'(8)(x-8)$.

$$\bullet f(x) = \sqrt[3]{x} = x^{1/3} \implies f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3(\sqrt[3]{x})^2}$$

$$\bullet f(8) = \sqrt[3]{8} = 2.$$

$$\bullet f'(8) = \frac{1}{3(\sqrt[3]{8})^2} = \frac{1}{3 \cdot 2^2} = \frac{1}{12}.$$

$$\text{So } L(x) = 2 + \frac{1}{12}(x-8).$$

$$L(8.2) = 2 + \frac{1}{12}(8.2-8)$$

$$= 2 + \frac{1}{12}(.2)$$

$$= 2 + \frac{1}{12} \cdot \frac{1}{5}$$

$$= \boxed{2 + \frac{1}{60}}.$$