MTH 161

Midterm ~~1~~ 2

Tuesday, November 8, 2016

NAME (please print legibly): Solutions

Your University ID Number: ______

Circle your instructor and class time:

Bobkova (MWF 9:00) Doyle (TR 9:40) Doyle (TR 3:25)
Lubkin (MW 2:00) Yamazaki (MWF 10:25)

Please read the following instructions very carefully:

- This exam is double-sided. Check that it has all 6 problems and 10 pages. There is an empty page at the end which can be used for scratch paper.
- Only pens/pencils are allowed. The presence of notes, calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.
- Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given. Clearly circle or label your final answers.
- Sign the following academic honesty statement: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.*

Signature: ______

QUESTION	VALUE	SCORE
1	20	
2	24	
3	14	
4	15	
5	15	
6	12	
TOTAL	100	

1. (20 points) Compute the following limits. You may ***not*** use L'Hôpital's rule.

(a) $\lim_{x\to 0} \frac{5x}{\tan(2x)}$ $= \lim_{x\to 0} 5x \cdot \frac{\cos(2x)}{\sin(2x)}$

$$= \lim_{x\to 0} \frac{5x}{2x} \cdot \frac{2x}{\sin(2x)} \cdot \cos(2x)$$

$$= \lim_{x\to 0} \frac{5}{2} \cdot \frac{2x}{\sin(2x)} \cdot \cos(2x)$$

$$= \frac{5}{2} \cdot 1 \cdot 1 = \boxed{\frac{5}{2}}$$

(b) $\lim_{x\to\infty} \frac{4x^3+2x+1}{2x^3-1}$ $= \lim_{x\to\infty} \frac{\frac{1}{x^3}(4x^3+2x+1)}{\frac{1}{x^3}(2x^3-1)}$

$$= \lim_{x\to\infty} \frac{4+\frac{2}{x^2}+\frac{1}{x^3}}{2-\frac{1}{x^3}}$$

$$= \frac{4+0+0}{2-0} = \frac{4}{2} = \boxed{2}$$

(c) $\lim_{x\to-\infty} \frac{3x^5-4x^4+2x-10}{2x^2-4x+1}$ $= \lim_{x\to-\infty} \frac{\frac{1}{x^2}(3x^5-4x^4+2x-10)}{\frac{1}{x^2}(2x^2-4x+1)}$

$$= \lim_{x\to-\infty} \frac{3x^3-4x^2+2-\frac{10}{x^2}}{2-\frac{4}{x}+\frac{1}{x^2}}.$$

The numerator tends to $-\infty$, and the denominator tends to 2.
Therefore the limit is $\boxed{-\infty}$

(d) $\lim_{x\to\infty} \sqrt{4x^2+1} - 2x$ $= \lim_{x\to\infty} \dfrac{(\sqrt{4x^2+1} - 2x)(\sqrt{4x^2+1} + 2x)}{\sqrt{4x^2+1} + 2x}$

$$= \lim_{x\to\infty} \frac{4x^2+1 - 4x^2}{\sqrt{4x^2+1} + 2x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{4x^2+1} + 2x}.$$

Since the denominator tends to ∞, the limit is $\boxed{0}$.

(e) Find all horizontal asymptotes of the function $f(x) = \dfrac{2e^x + 8}{3e^x - 4}$.

$$\lim_{x\to\infty} \frac{2e^x+8}{3e^x-4} = \lim_{x\to\infty} \frac{\frac{1}{e^x}(2e^x+8)}{\frac{1}{e^x}(3e^x-4)} = \lim_{x\to\infty} \frac{2+\frac{8}{e^x}}{3-\frac{4}{e^x}}$$

$$= \frac{2+0}{3-0} = \boxed{\frac{2}{3}}$$

$$\lim_{x\to-\infty} \frac{2e^x+8}{3e^x-4} = \frac{2\cdot 0+8}{3\cdot 0-4} = \frac{8}{-4} = \boxed{-2}$$

Horizontal asymptotes: $\boxed{y = \frac{2}{3} \quad \text{and} \quad y = -2}$

2. (24 points) For each of the following functions y, compute the derivative $\frac{dy}{dx}$.

(a) $y = 4x^5 - 16\sqrt[5]{x^3} + \frac{7}{\sqrt{x}}$ $= 4x^5 - 16x^{3/5} + 7x^{-1/2}$

$$\frac{dy}{dx} = \boxed{20x^4 - \frac{3 \cdot 16}{5}x^{-2/5} - \frac{7}{2}x^{-3/2}}$$

(b) $y = (4x^3 + 2x)(e^x + x^e)$

$$\frac{dy}{dx} = \boxed{(12x^2+2)(e^x+x^e) + (4x^3+2x)(e^x+ex^{e-1})}$$

(c) $y = \dfrac{\tan(x)}{e^x} + \cos(\ln(x^2+1)))$

$$\frac{dy}{dx} = \boxed{\frac{e^x \cdot \sec^2(x) - \tan(x) \cdot e^x}{(e^x)^2} - \sin\left(\ln(x^2+1)\right) \cdot \frac{1}{x^2+1} \cdot 2x}$$

(d) $y = \tan^{-1}(x^3)$

$$\frac{dy}{dx} = \frac{1}{1+(x^3)^2} \cdot 3x^2 = \boxed{\frac{3x^2}{1+x^6}}$$

(e) Let $y = \dfrac{x^3 e^x}{\sqrt[4]{x+1}}$, and use logarithmic differentiation to compute $\dfrac{dy}{dx}$.

$$\ln(y) = \ln\left(\frac{x^3 e^x}{(x+1)^{1/4}}\right) = \ln(x^3) + \ln(e^x) - \ln\left((x+1)^{1/4}\right)$$

$$\ln(y) = 3\ln(x) + x - \frac{1}{4}\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = 3 \cdot \frac{1}{x} + 1 - \frac{1}{4} \cdot \frac{1}{x+1} \cdot 1$$

$$\frac{dy}{dx} = y\left(\frac{3}{x} + 1 - \frac{1}{4(x+1)}\right)$$

$$= \boxed{\frac{x^3 e^x}{\sqrt[4]{x+1}}\left(\frac{3}{x} + 1 - \frac{1}{4(x+1)}\right)}$$

(f) $y = (4x+1)^{4x+1}$

$$\ln(y) = \ln\left((4x+1)^{4x+1}\right)$$

$$\ln(y) = (4x+1)\ln(4x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = 4 \cdot \ln(4x+1) + (4x+1) \cdot \frac{1}{4x+1} \cdot 4$$

$$\frac{dy}{dx} = y\left(4\ln(4x+1) + 4\right)$$

$$= \boxed{(4x+1)^{4x+1}\left(4\ln(4x+1) + 4\right)}.$$

3. (14 points) For each of the following equations, find $\frac{dy}{dx}$.

(a) $y = \sin(x + y)$

$$\frac{dy}{dx} = \cos(x+y) \cdot \left[1 + \frac{dy}{dx}\right]$$

$$\frac{dy}{dx} = \cos(x+y) + \cos(x+y)\frac{dy}{dx}$$

$$\frac{dy}{dx} - \cos(x+y)\frac{dy}{dx} = \cos(x+y)$$

$$\left(1 - \cos(x+y)\right)\frac{dy}{dx} = \cos(x+y)$$

$$\frac{dy}{dx} = \boxed{\frac{\cos(x+y)}{1-\cos(x+y)}}$$

(b) $e^{xy} = x^2 + y^2$

$$e^{xy} \cdot \left[1 \cdot y + x \cdot \frac{dy}{dx}\right] = 2x + 2y\frac{dy}{dx}$$

$$ye^{xy} + xe^{xy}\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$$

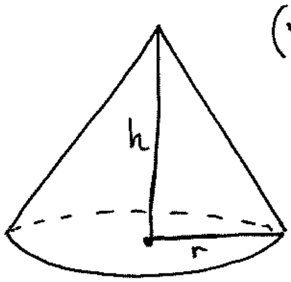
$$xe^{xy}\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - ye^{xy}$$

$$(xe^{xy} - 2y)\frac{dy}{dx} = 2x - ye^{xy}$$

$$\frac{dy}{dx} = \boxed{\frac{2x - ye^{xy}}{xe^{xy} - 2y}}$$

4. (15 points) Sand is being dumped into a pile at a rate of 1600 ft^3/min. The pile is in the shape of a cone whose height is always half of its base radius. How fast is the height of the pile increasing when the height is 10 ft?

(**Hint:** The volume of a cone of base radius r and height h is $V = \frac{\pi}{3}r^2h$.)



(V = volume)

WANT: $\left.\frac{dh}{dt}\right|_{h=10}$

KNOW: $\frac{dV}{dt} = 1600$

$V = \frac{\pi}{3}r^2h$

$h = \frac{1}{2}r.$

$r = 2h$

$V = \frac{\pi}{3}r^2h.$

$V = \frac{\pi}{3}(2h)^2h$

$V = \frac{\pi}{3}\cdot 4h^3$

$V = \frac{4\pi}{3}h^3$

$\frac{dV}{dt} = \frac{4\pi}{3}\cdot 3h^2\cdot\frac{dh}{dt}$

when $h = 10$:

$1600 = \frac{4\pi}{3}\cdot 3\cdot 10^2\cdot\frac{dh}{dt}$

$1600 = 4\pi\cdot 100\,\frac{dh}{dt}$

$1600 = 400\pi\,\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1600}{400\pi}$

$= \boxed{\frac{4}{\pi}\ \text{ft/min}}$

5. (15 points) A particle is moving in a straight line. Its position (in feet) at time t seconds is given by the function

$$x(t) = t^3 - 9t^2 + 24t + 8, \ t \geq 0.$$

(a) Find the velocity and acceleration of the particle at time t.

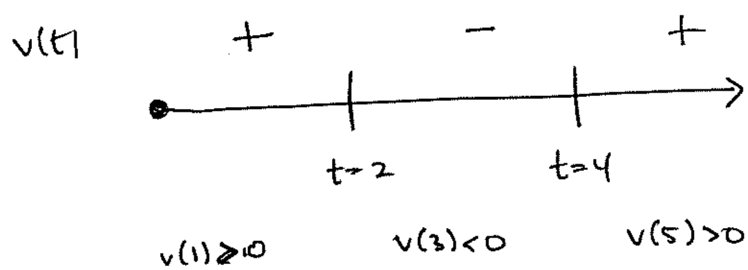
$$v(t) = x'(t) = 3t^2 - 18t + 24$$

$$a(t) = v'(t) = 6t - 18$$

(b) On what interval(s) is the particle moving forward? moving backward?

$$v(t) = 0 \Rightarrow 3(t^2 - 6t + 8) = 0 \Rightarrow 3(t-2)(t-4) = 0$$

$$\Rightarrow t = 2, \ t = 4.$$



Forward: $[0, 2) \cup (4, \infty)$

Backward: $(2, 4)$

(c) What is the **total** distance the particle travels in the first 5 seconds?

$$|x(2) - x(0)| + |x(4) - x(2)| + |x(5) - x(4)|$$

$$= |2^3 - 9 \cdot 2^2 + 24 \cdot 2 + 8 - 8| + |4^3 - 9 \cdot 4^2 + 24 \cdot 4 + 8 - (2^3 - 9 \cdot 2^2 + 24 \cdot 2 + 8)|$$

$$+ |5^3 - 9 \cdot 5^2 + 24 \cdot 5 + 8 - (4^3 - 9 \cdot 4^2 + 24 \cdot 4 + 8)|$$

$$= |20| + |-4| + |4| = 20 + 4 + 4 = \boxed{28 \text{ ft}}$$

6. (12 points) An object with a temperature of 50°F is placed into an oven with a temperature of 350°F. After 10 minutes, the object's temperature has increased to 150°F. (You do not have to simplify your answers.)

(a) What is the object's temperature after 20 minutes?

(surrounding temperature)

$$\underline{T(t) = 350 + Ce^{kt}}$$

- $T(0) = 50$

$$350 + Ce^{k0} = 50$$
$$350 + C = 50$$
$$C = -300$$

$$\underline{T(t) = 350 - 300e^{kt}}$$

- $T(10) = 150$

$$350 - 300e^{k\cdot 10} = 150$$
$$-300e^{10k} = -200$$
$$e^{10k} = \frac{2}{3} \Rightarrow 10k = \ln(2/3) \Rightarrow k = \frac{1}{10}\ln(2/3)$$

$$\underline{T(t) = 350 - 300e^{\frac{1}{10}\ln(\frac{2}{3})t}}$$

$$T(20) = 350 - 300e^{\frac{1}{10}\ln(\frac{2}{3})\cdot 20}$$
$$= \boxed{\left(350 - 300e^{2\ln(\frac{2}{3})}\right)^\circ F}$$

NOTE: This can be simplified:

$$350 - 300\left(\frac{2}{3}\right)^2$$
$$= 350 - 300\cdot\frac{4}{9}$$
$$= 350 - \frac{400}{3}$$
$$= \frac{650}{3} = \boxed{216.\overline{6}\ ^\circ F}$$

(b) How long will it take for the object to reach a temperature of 300°F?

$$T(t) = 300$$
$$350 - 300e^{\frac{1}{10}\ln(2/3)t} = 300$$
$$-300e^{\frac{t}{10}\ln(2/3)} = -50$$
$$e^{\frac{t}{10}\ln(2/3)} = \frac{1}{6}$$
$$\frac{t}{10}\ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{6}\right)$$
$$t = \boxed{\frac{10\ln(\frac{1}{6})}{\ln(\frac{2}{3})} \text{ minutes}}$$