MTH 161 Midterm 1 October 13, 2015

Please print your name and student ID again, and circle your instructor's name:

Name (please print): Solutions

University student ID: ____________________

Bobkova (TR 9:40) Bridy (MW 2:00) Doyle (MWF 10:25)

Hambrook (TR 3:25) Lubkin (MWF 9:00) Murphy (TR 4:50)

Please read the following instructions:

Only pens/pencils and a single 3 in. × 5 in. index card with formulas are allowed. The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.

Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given. Clearly circle or label your final answers.

QUESTION	VALUE	SCORE
1	12	
2	18	
3	25	
4	15	
5	20	
6	10	
TOTAL	100	

1. (12 points) Let $f(x) = \sqrt{5 + x^2}$ and $g(x) = \dfrac{2x}{3x - 2}$.

(a) Evaluate $(f \circ g)(2)$. $f(g(2)) = f\left(\frac{2\cdot 2}{3\cdot 2-2}\right) = f\left(\frac{4}{4}\right)$

$= f(1)$

$= \sqrt{5+1^2} = \boxed{\sqrt{6}}$

(b) Evaluate $(g \circ f)(2)$. $g(f(2)) = g\left(\sqrt{5+2^2}\right) = g(\sqrt{9})$

$= g(3)$

$= \frac{2\cdot 3}{3\cdot 3-2} = \boxed{\frac{6}{7}}$

(c) Find a formula for $g^{-1}(x)$, the inverse of g.

$y = g(x)$

$y = \frac{2x}{3x-2}$

$y(3x-2) = 2x$

$3xy - 2y = 2x$

$3xy - 2x = 2y$

$x(3y-2) = 2y$

$x = \frac{2y}{3y-2} \rightsquigarrow \boxed{g^{-1}(x) = \frac{2x}{3x-2}}$

2. (18 points)

(a) Solve for x:

$$2\ln(x) = \ln(2x - 1)$$

$$\ln(x^2) = \ln(2x-1)$$

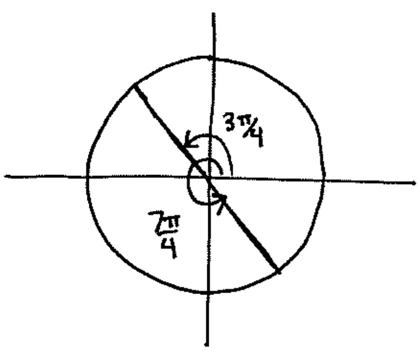
$$x^2 = 2x-1$$

$$x^2-2x+1 = 0$$

$$(x-1)^2 = 0$$

$$\boxed{x = 1}$$

(b) Find all values of x in the interval $[0, 4\pi]$ that satisfy the following equation:



$$\sin(x) + \cos(x) = 0$$

$$\sin(x) = -\cos(x)$$

$$\tan(x) = -1.$$

The solutions to $\tan(x) = -1$ between 0 and 2π are

$$\frac{3\pi}{4}, \frac{7\pi}{4}.)$$

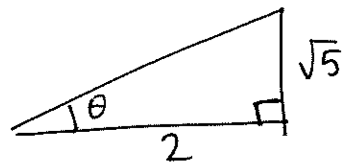
so the solutions between 0 and 4π are

$$\left\{ \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{3\pi}{4}+2\pi, \frac{7\pi}{4}+2\pi \right\}$$

$$= \boxed{\left\{ \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4} \right\}}$$

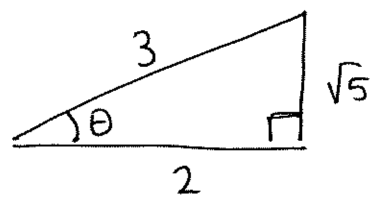
(c) Find $\cos\left(\tan^{-1}\left(\frac{\sqrt{5}}{2}\right)\right)$.

Let $\theta = \tan^{-1}\left(\frac{\sqrt{5}}{2}\right)$. Then consider the following right triangle:



By the Pythagorean theorem, the hypotenuse has length

$$\sqrt{(\sqrt{5})^2 + 2^2} = \sqrt{5+4} = \sqrt{9} = 3;$$



,

so $\cos\theta = \boxed{\frac{2}{3}}$.

3. (25 points) Compute the following limits. If the limit is infinite, write "$-\infty$" or "$+\infty$" as appropriate. If the limit does not exist and is not $+\infty$ or $-\infty$, write "DNE." *You may only use methods discussed in this class thus far.*

(a) $\lim_{x \to 1} \frac{3x^2 - x + 1}{x^2 - 5} = \frac{3(1)^2 - 1 + 1}{(1)^2 - 5} = \frac{3}{-4} = \boxed{-\frac{3}{4}}$

(b) $\lim_{x \to 1^-} \frac{x^2 - 1}{x^2 - 2x + 1} = \lim_{x \to 1^-} \frac{(x-1)(x+1)}{(x-1)^2}$

$= \lim_{x \to 1^-} \frac{x+1}{x-1}.$

This looks like "$\frac{2}{0}$," which suggests the limit may be infinite. Since the limit is as $x \to 1$ from the left, we are only interested in values of x less then 1. If x is less then 1 and close to 1, then

- $x+1 > 0$
- $x-1 < 0$

$\Big\}$ so $\frac{x+1}{x-1} < 0.$

Therefore $\lim_{x \to 1^-} \frac{x+1}{x-1} = \boxed{-\infty}$

(c) $\lim_{x\to -2} \frac{x+2}{|x+2|} =$

• if $x < -2$, then $|x+2| = -(x+2)$, so

$$\lim_{x\to -2^-} \frac{x+2}{|x+2|} = \lim_{x\to -2^-} \frac{x+2}{-(x+2)} = \lim_{x\to -2^-} -1 = -1$$

• if $x > 2$, then $|x+2| = x+2$, so

$$\lim_{x\to -2^+} \frac{x+2}{|x+2|} = \lim_{x\to -2^+} \frac{x+2}{x+2} = \lim_{x\to -2^+} 1 = 1.$$

Since the one-sided limits disagree, the limit $\boxed{\text{DNE}}$.

(d) $\lim_{x\to\infty} \frac{3x^4 - 4x^2 + 3}{-2x^5 + 2x - 4} = \lim_{x\to\infty} \frac{\frac{1}{x^5}(3x^4 - 4x^2 + 3)}{\frac{1}{x^5}(-2x^5 + 2x - 4)}$

$$= \lim_{x\to\infty} \frac{3/x - 4/x^3 + 3/x^5}{-2 + 2/x^4 - 4/x^5}$$

$$= \frac{0}{-2}$$

$$= \boxed{0}$$

(e) $\lim_{x\to\infty} \frac{4e^{2x}-8}{2e^x+8} = \lim_{x\to\infty} \frac{\frac{1}{e^x}(4e^{2x}-8)}{\frac{1}{e^x}(2e^x+8)}$

$$= \lim_{x\to\infty} \frac{4e^x - 8/e^x}{2 + 8/e^x}$$

$$= \lim_{x\to\infty} \frac{4e^x}{2}$$

$$= \boxed{+\infty}$$

(f) Find all horizontal asymptotes of the function $f(x) = \frac{\sqrt{x^2+1}+x}{x}$.

Rewrite $f(x) = \frac{\sqrt{x^2+1}}{x} + 1$.

- if $x<0$, then $\frac{\sqrt{x^2+1}}{x} = \frac{\sqrt{x^2+1}}{-|x|} = -\frac{\sqrt{x^2+1}}{|x|} = -\sqrt{\frac{x^2+1}{x^2}}$

$$= -\sqrt{1+\frac{1}{x^2}}.$$

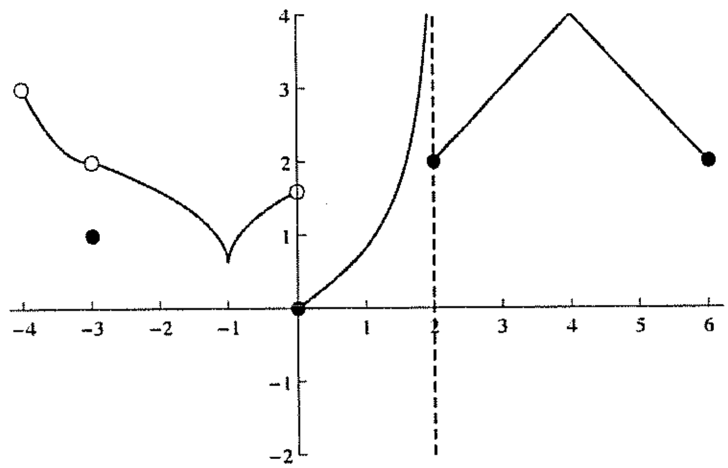
So $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left[-\sqrt{1+\frac{1}{x^2}}+1\right] = -1+1 = \boxed{0}$

- if $x>0$, then $\frac{\sqrt{x^2+1}}{x} = \frac{\sqrt{x^2+1}}{|x|} = \sqrt{\frac{x^2+1}{x^2}} = \sqrt{1+\frac{1}{x^2}}$.

So $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\sqrt{1+\frac{1}{x^2}}+1\right] = 1+1 = \boxed{2}$

Horizontal asymptotes: $\boxed{y=0 \text{ and } y=2}$

4. **(15 points)** Answer the following questions about the function $f(x)$ graphed below. You do not need to justify your answers.



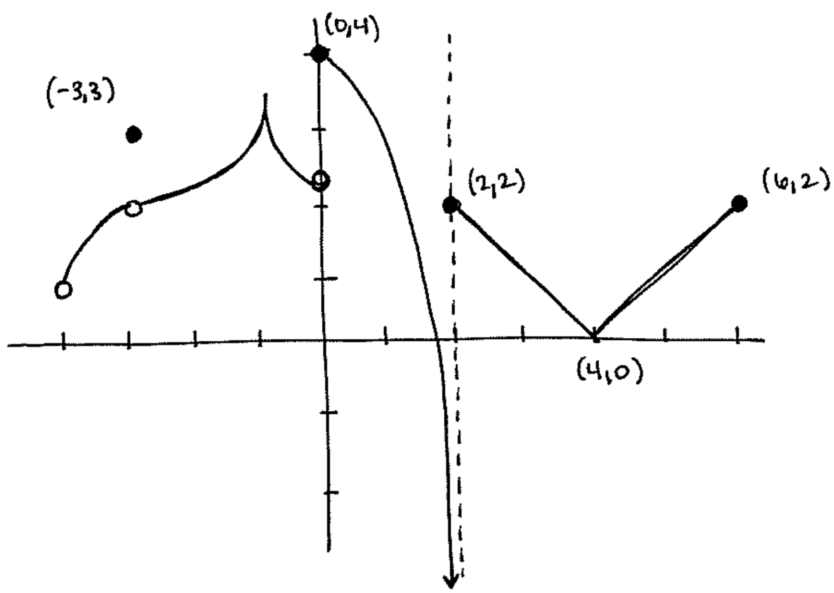
(a) For which values of x in the interval $(-4,6)$ is $f(x)$ *not* continuous?

$\boxed{-3, 0, 2}$

(b) For which values of x in the interval $(-4,6)$ is $f(x)$ *not* differentiable?

$\boxed{-3, -1, 0, 2, 4}$

(c) Sketch the graph of the function $g(x) = -f(x) + 4$ below. Label at least three points on the graph. [Reflect the original graph over the x-axis, then shift up 4.]



5. (20 points)

(a) Complete the definition of the derivative of a function $f(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(b) Use the definition of the derivative to show that the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.

$$\frac{d}{dx}\sqrt{x} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(c) Using that the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$, give an equation for the tangent line to the graph of $y = \sqrt{x}$ at $x = 4$.

$$\left.\begin{array}{l} \text{point: } (4, \sqrt{4}) = (4, 2) \\ \text{slope: } \frac{1}{2\sqrt{4}} = \frac{1}{4}. \end{array}\right\}$$

$$\boxed{y - 2 = \frac{1}{4}(x - 4)}$$

6. (10 points)

(a) State the Intermediate Value Theorem.

If $f(x)$ is continuous on the interval $[a,b]$, and if y_0 is any number between $f(a)$ and $f(b)$, then there is some c in the interval (a,b) such that $f(c) = y_0$.

(b) Show that the equation $e^x + 3x = x^2 + 2$ has a solution x between 0 and 1. Show your work.

Let $f(x) = e^x + 3x - x^2 - 2$.

- Since f is a sum of a polynomial and an exponential function, f is continuous everywhere; in particular, f is continuous on $[0,1]$.
- $f(0) = e^0 + 3(0) - (0)^2 - 2 = 1 - 2 = -1$.
- $f(1) = e^1 + 3(1) - (1)^2 - 2 = e + 3 - 1 - 2 = e$
- Since $-1 < 0 < e$, there is some c in $(0,1)$ for which

$$f(c) = 0.$$

$$\Rightarrow e^c + 3c - c^2 - 2 = 0$$

$$\Rightarrow e^c + 3c = c^2 + 2,$$

so the equation $e^x + 3x = x^2 + 2$ has a solution between 0 and 1.