MTH 161

Final Exam
Sunday, December 17, 2017

Last Name (Family Name) Solutions
First Name (Given Name) ______
Student ID Number: ______
Circle your instructor and class time:

Hambrook (MW 10:25) Hambrook (MW 2:00)
Lorman (TuTh 9:40) Lubkin (MW 9:00) Xi (TuTh 3:25)

Please read the following instructions very carefully:

- Only pens and pencils are allowed. The presence of notes, calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.
- This exam has 15 problems and 24 pages. Check that your exam is complete when you start.
- Show your work. You may not receive full credit if insufficient justification is given.
- Clearly circle or label your final answers.
- If you need extra space, use the back of the opposite page, and write that you are doing so.
- Sign the following academic honesty statement: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.*

Signature: ______

Part A

QUESTION	VALUE	SCORE
1	20	
2	16	
3	16	
4	12	
5	14	
6	14	
7	8	
TOTAL	100	

Part B

QUESTION	VALUE	SCORE
8	8	
9	12	
10	8	
11	15	
12	20	
13	10	
14	12	
15	15	
TOTAL	100	

Part A

1. **(20 points)**

(a) If $f(x) = \dfrac{x+1}{2x+1}$, find a formula for $f^{-1}(x)$.

Let $y = \dfrac{x+1}{2x+1}$

$2xy + y = x+1$

$(2y-1)x = 1-y$

$x = \dfrac{y-1}{1-2y}$

$f^{-1}(x) = \dfrac{x-1}{1-2x}$, $2x \neq 1$, $x \neq \frac{1}{2}$.

(b) Solve $\ln x + \ln(x-1) = \ln 6$

$x > 0$, $x-1 > 0$

~~$\ln \frac{x}{1}$~~ $\ln x \cdot (x-1) = \ln 6$

$x \cdot (x-1) = 6.$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$, since $x > 0$.

So $x = 3$.

(c) Find the exact value of $\tan\left(\sin^{-1}\left(\frac{-1}{9}\right)\right)$.

Let $\theta = \sin^{-1}\left(-\frac{1}{9}\right)$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. but $-\frac{1}{9} < 0$

$\theta \in \left[-\frac{\pi}{2}, 0\right)$.

$\cos\theta = \sqrt{1-\frac{1}{81}} = \frac{\sqrt{80}}{9}$. $\cos\theta > 0$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{1}{9}}{\frac{\sqrt{80}}{9}} = -\frac{1}{\sqrt{80}}$$

(d) Solve $|x| + |2x-1| \geq 7$.

① If $x \leq 0$.

$-x + 1 - 2x \geqslant 7$

$x \leq -2$

② if $0 < x \leq \frac{1}{2}$

$x + 1 - 2x \geqslant 7$

$x \leq -6$ Impossible

③ If $x > \frac{1}{2}$

$x + 2x - 1 \geqslant 7$

$x \geqslant \frac{8}{3}$

So $x \in (-\infty, -2] \cup [\frac{8}{3}, \infty)$.

2. (16 points) Compute the derivative (with respect to x) of each of the following functions.

(a) $\dfrac{x^3}{\cos(x^3)}$

$$\frac{d}{dx}\left(\frac{x^3}{\cos x^3}\right)$$

$$= \frac{3x^2}{\cos(x^3)} + \frac{-x^3}{(\cos(x^3))^2} \cdot (-\sin(x^3)\cdot 3x^2.$$

$$= \frac{3x^2}{\cos(x^3)} + \frac{3x^5 \sin(x^3)}{\cos(x^3)}$$

(b) $e^{\sqrt{\ln x}}$

$$\frac{d}{dx}\left(e^{\sqrt{\ln x}}\right)$$

$$= e^{\sqrt{\ln x}} \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x}.$$

(c) $\sqrt[3]{1+x^2}+\dfrac{1}{\sqrt{1+x^3}}$

$$\frac{d}{dx}\left(\sqrt[3]{1+x^2}+\frac{1}{\sqrt{1+x^3}}\right)$$

$$=\frac{d}{dx}\left((1+x^2)^{\frac{1}{3}}+(1+x^3)^{-\frac{1}{2}}\right)$$

$$=\frac{1}{3}(1+x^2)^{-\frac{2}{3}}\cdot 2x-\frac{1}{2}(1+x^3)^{-\frac{3}{2}}\cdot 3x^2.$$

(d) $(\ln x)^{\ln x}$

$$\frac{d}{dx}\left[(\ln x)^{\ln x}\right]$$

$$=\frac{d}{dx}\quad e^{\ln x\cdot\ln(\ln x)}$$

$$=e^{\ln x\cdot\ln(\ln x)}\cdot\left(\frac{1}{x}\cdot\ln(\ln x)+\ln x\cdot\frac{1}{\ln x}\cdot\frac{1}{x}\right).$$

$$=\frac{1}{x}(\ln x)^{\ln x}\cdot\left(\ln(\ln x)+1\right)$$

3. (16 points) Evaluate the following limits.

(a) $\lim_{x \to 1^+} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

As $x \to 1^+$, $x^2 - 9 \to -8$

while $x^2 + 2x - 3 = (x+3)(x-1) \to 0^+$

So $\lim_{x \to 1^+} \dfrac{x^2 - 9}{x^2 + 2x - 3} = -\infty$.

(b) $\lim_{x \to -\infty} \dfrac{\sqrt{x^6 + 1}}{(2x+1)^3}$ $\quad$ $x \to -\infty$, $-x > 0$.

$$= \lim_{x \to -\infty} \frac{\dfrac{\sqrt{x^6+1}}{(-x)^3}}{\dfrac{(2x+1)^3}{(-x)^3}} = \lim_{x \to -\infty} \frac{\sqrt{1 + \frac{1}{x^6}}}{-\left(2 + \frac{1}{x}\right)^3} = -\frac{1}{8}$$

(c) $\lim_{x\to 0} \frac{\sqrt{ax+b^2}-b}{x}$ (where a and b are positive constants)

$$= \lim_{x\to 0} \frac{ax+b^2-b^2}{x\,(\sqrt{ax+b^2}+b)} = \lim_{x\to 0} \frac{a}{\sqrt{ax+b^2}+b} = \frac{a}{2b}.$$

(d) $\lim_{x\to 0} \frac{|2x-1|-|2x+1|}{x}$ as $x \to 0$, $2x-1 < 0$, $2x+1>0$.

$$= \lim_{x\to 0} \frac{1-2x-2x-1}{x} = -4.$$

4. **(12 points)** Consider the curve $y \sin 2x = x \cos 2y$.

(a) Find $\frac{dy}{dx}$ at the point $(\pi/2, \pi/4)$.

Taking derivative on both side

$$y' \sin 2x + 2y \cos 2x = \cos 2y + 2x(-\sin 2y)y'.$$

$$y'(\sin 2x + 2x \sin 2y) = \cos 2y - 2y \cos 2x.$$

when $x = \frac{\pi}{2}$, $y = \frac{\pi}{4}$, this gives

$$y' \cdot \left(\sin \pi + \pi \sin \frac{\pi}{2}\right) = \cos \frac{\pi}{2} - \frac{\pi}{2} \cos \pi.$$

$$\pi y' = \frac{\pi}{2}$$

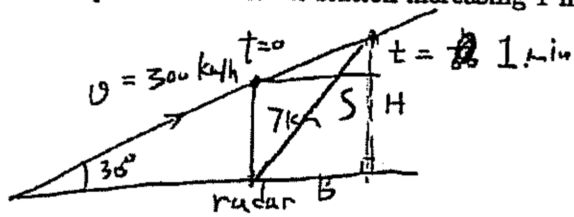
$$\boxed{y' = \frac{1}{2}}$$

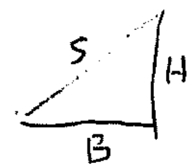
(b) Find an equation for the tangent line to the curve at the point $(\pi/2, \pi/4)$.

Slope $m = \frac{1}{2}$

$$y - \frac{\pi}{4} = \frac{1}{2}\left(x - \frac{\pi}{2}\right).$$

5. (14 points) A plane is climbing at an angle of 30° while flying at a constant speed of 300 km/h. It passes over a ground radar station at an altitude of 7 km. At what rate is the distance from the plane to the radar station increasing 1 minute later?





$v = 5$ km/min

Solution: $S(t) = \sqrt{H^2 + B^2}$

$$= \sqrt{\left(7 + \frac{300t}{60}\cdot \sin 30^\circ\right)^2 + \left(\frac{300t}{60}\cdot \cos 30^\circ\right)^2}$$

$$= \sqrt{\left(7 + \frac{5}{2}t\right)^2 + \left(\frac{5}{2}\sqrt{3}\,t\right)^2}$$

$$= \sqrt{49 + 35t + \frac{25}{4}t^2 + \frac{75}{4}t^2}$$

$$= \sqrt{25t^2 + 35t + 49}$$

$$S'(t) = \frac{50t + 35}{2\sqrt{25t^2 + 35t + 49}}$$

$$S'(1) = \frac{85}{2\sqrt{109}}$$

6. (14 points) The height of the foam in a glass of (root) beer decreases at a rate proportional to the current height. The glass is filled with (root) beer so that the top 5 cm is foam. After 60 seconds, only 2 cm of foam remains.

(a) Find an expression for the height of the foam t seconds after the (root) beer is poured.

height : $H(t)$.

$$\frac{dH}{dt} = kH \qquad (k<0).$$

$$H = Ce^{kt}. \qquad \text{at } t=0,\ H(0) = 5.$$

$$H = 5e^{kt}.$$

$$H(60) = 5e^{k60} = 2.$$

$$e^{60k} = \frac{2}{5}$$

$$60k = \ln\frac{2}{5}$$

$$k = \frac{1}{60}\ln\frac{2}{5}$$

$$H(t) = 5\,e^{\ln\frac{2}{5}\cdot\frac{t}{60}}$$

$$= 5\cdot\left(\frac{2}{5}\right)^{\frac{t}{60}}$$

(b) At what time is the height of the foam 4 cm?

$$5\left(\frac{2}{5}\right)^{\frac{t}{60}} = 4.$$

$$\left(\frac{2}{5}\right)^{\frac{t}{60}} = \frac{4}{5}.$$

$$\frac{t}{60} = \frac{\ln\frac{4}{5}}{\ln\frac{2}{5}}$$

$$t = 60\cdot\frac{\ln\frac{4}{5}}{\ln\frac{2}{5}}.$$

(c) How long must we wait until the foam completely disappears?

Never completely disappears according to this model.

7. (8 points) For each statement below, circle T (true) if the statement is always true. Otherwise, circle F (false)

(a) T or F If f is continuous on (a, b) and $f(a) < 0 < f(b)$, then there is a number c in (a, b) such that $f(c) = 0$.

(b) T or F If $f'(0) = 5$, then $\lim_{h \to 0} \frac{f(h)}{h} = 5$.

(c) T or F If $f(1) = g(1)$ and $f'(x) \leq g'(x)$ for all x in $[0, 1]$, then $f(0) \geq g(0)$.

(d) T or F $f(x) = x|x|$ is differentiable at every real number x.

Part B

8. (8 points) For each statement below, circle T (true) if the statement is always true. Otherwise, circle F (false)

(a) T or F If f is continuous, then $\int_0^5 f(x)dx \leq \int_0^5 |f(x)|dx$

(b) T or F If f is continuous, then $\int_0^5 f(x)dx \leq \int_0^5 (f(x))^2 dx$

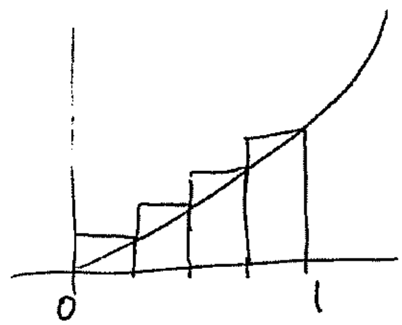
(c) T or F If f is continuous, then $\int_0^5 f(x)dx \leq \int_0^5 (f(x)+2)dx$

(d) T or F If f is continuous, then $\int_0^5 f(x)dx \leq \int_0^{10} f(x)dx$

9. (12 points)

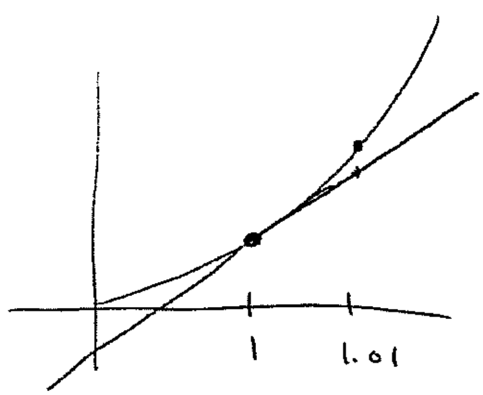
(a) Let $f(x)$ be an increasing function. If a Riemann sum with right endpoints is used to approximate $\int_0^1 f(x)dx$, must the Riemann sum be larger than the integral? Justify your answer with an appropriate sketch.

Yes.



(b) Let $f(x)$ be an increasing function and let $L(x) = f'(1)(x-1) + f(1)$ be the linear approximation function of $f(x)$ at 1. Must $L(1.01)$ be larger than $f(1.01)$? Justify your answer with an appropriate sketch.

No, smaller.



10. (8 points) Evaluate the following limits.

(a) $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x} \cdot \frac{(\sqrt{1+2x}+\sqrt{1-4x})}{(\sqrt{1+2x}+\sqrt{1-4x})}$

$$= \lim_{x\to 0} \frac{1+2x-(1-4x)}{x(\sqrt{1+2x}+\sqrt{1-4x})} = \lim_{x\to 0} \frac{6\cancel{x}}{\cancel{x}(\sqrt{1+2x}+\sqrt{1-4x})}$$

$$= \lim_{x\to 0} \frac{6}{(\underbrace{\sqrt{1+2x}}_{\downarrow\, 1}+\underbrace{\sqrt{1-4x}}_{\downarrow\, 1})} = \frac{6}{2} = \boxed{3}$$

(b) $\lim_{x\to 0^+} (\tan 2x)^x = \lim_{x\to 0^+} e^{\ln(\tan 2x)^x} = e^{\lim_{x\to 0^+} x\ln(\tan(2x))} = \boxed{e^0 = 1}$

by L'Hopital's rule

$$\lim_{x\to 0^+} x\ln(\tan(2x)) = \lim_{x\to 0^+} \frac{\overbrace{\ln(\tan(2x))}^{\to -\infty}}{\underbrace{1/x}_{\downarrow\,\infty}} = \lim_{x\to 0^+} \frac{2\sec^2(2x)}{\tan(2x)\left(-\frac{1}{x^2}\right)}$$

$$= \lim_{x\to 0} \frac{-2x^2\cancel{\cos(2x)}}{\sin(2x)\cos^{\cancel{2}}(2x)} = \lim_{x\to 0^+} \frac{-2x^2}{\sin(2x)\cos(2x)} = \lim_{x\to 0^+} \frac{\overbrace{-2x^2}^{\to 0}}{\underbrace{\frac{1}{2}\sin(4x)}_{\downarrow\, 0}}$$

L'Hopital again

$$= \lim_{x\to 0^+} \frac{-8x}{4\cos(4x)} = \boxed{0}$$

11. (15 points) Evaluate the following integrals.

(a) $\int_1^{e^2} \frac{\sqrt{\ln x + 1}}{x} dx$

Substitute $u = \ln x + 1$

$du = \frac{1}{x} dx$

$u(1) = \ln(1) + 1 = 1$

$u(e^2) = \ln(e^2) + 1 = 3$

$$= \int_1^3 \sqrt{u}\, du = \frac{2}{3} u^{3/2} \Big|_{u=1}^{u=3}$$

$$= \frac{2}{3}\left(3^{3/2} - 1^{3/2}\right) = \boxed{\frac{2}{3}\left(3^{3/2} - 1\right)}$$

(b) $\int \frac{x^5}{\sqrt{x^3+1}} dx$

$u = x^3 + 1 \rightsquigarrow x^3 = u - 1$

$du = 3x^2\, dx$

$$= \int \frac{x^2 \cdot x^3}{\sqrt{x^3+1}} dx = \frac{1}{3}\int \frac{u-1}{\sqrt{u}} du = \frac{1}{3}\int \frac{u}{\sqrt{u}} du - \frac{1}{3}\int \frac{1}{\sqrt{u}} du$$

$$= \frac{1}{3}\int \sqrt{u}\, du - \frac{1}{3}\int \frac{1}{\sqrt{u}} du = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} - \frac{1}{3} \cdot 2\sqrt{u} + C$$

$$= \boxed{\frac{2}{9}(x^3+1)^{3/2} - \frac{2}{3}\sqrt{x^3+1} + C}$$

(c) $\int_0^3 |e^x - 2|dx$

$$= \int_0^{\ln(2)} -e^x+2\,dx + \int_{\ln(2)}^3 e^x-2\,dx$$

$$= \left(-e^x+2x\right)\Big|_0^{\ln(2)} + \left(e^x-2x\right)\Big|_{\ln(2)}^3$$

$$= -e^{\ln(2)} + 2\ln(2) + e^0 - 2\cdot 0$$
$$+ e^3 - 2\cdot 3 - e^{\ln(2)} + 2\ln(2)$$

$$= -2 + 4\ln(2) + 1 + e^3 - 6 - 2$$

$$= \boxed{-9 + 4\ln(2) + e^3}$$

$e^x - 2 > 0$

$e^x > 2$

$x > \ln(2)$

$$|e^x-2| = \begin{cases} e^x-2 & \ln(2) \le x \le 3 \\ -(e^x-2) & 0 \le x < \ln(2) \end{cases}$$

12. (20 points) Consider the function f with its first and second derivatives:

$$f(x) = \frac{1}{\sqrt[3]{x^3+1}}, \quad f'(x) = \frac{-x^2}{\left(\sqrt[3]{x^3+1}\right)^4}, \quad f''(x) = \frac{2x(x^3-1)}{\left(\sqrt[3]{x^3+1}\right)^7}.$$

(a) Find the domain of $f(x)$.

$\sqrt[3]{x^3+1} = 0$ when $x^3+1=0$ when $x^3=-1$ when $x=-1$.

Domain: all real numbers except $x=-1$

$= (-\infty, -1) \cup (-1, \infty)$.

(b) List all x-intercepts and y-intercepts of $f(x)$.

x-intercepts:

$f(x)=0$

$\frac{1}{\sqrt[3]{x^3+1}} = 0$ no solutions.

$\boxed{\text{no } x\text{-intercepts.}}$

y-intercepts:

$f(0) = \frac{1}{\sqrt[3]{0^3+1}} = 1$. $\boxed{y\text{-intercept: } (0,1)}$

Reminder: $f(x) = \dfrac{1}{\sqrt[3]{x^3+1}}$, $f'(x) = \dfrac{-x^2}{\left(\sqrt[3]{x^3+1}\right)^4}$, $f''(x) = \dfrac{2x(x^3-1)}{\left(\sqrt[3]{x^3+1}\right)^7}$.

(c) Compute $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ for any vertical asymptotes $x = a$.

Only possible vertical asymptote is at $x = -1$

$$\lim_{x\to -1^-} \frac{1}{\sqrt[3]{x^3+1}} = -\infty$$

$$\lim_{x\to -1^+} \frac{1}{\sqrt[3]{x^3+1}} = \infty$$

(d) Compute $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to\infty} f(x)$. List all horizontal asymptotes of $f(x)$.

$$\lim_{x\to -\infty} \frac{1}{\sqrt[3]{x^3+1}} = 0$$

$$\lim_{x\to \infty} \frac{1}{\sqrt[3]{x^3+1}} = 0$$

Horizontal asymptote: $y = 0$

Reminder: $f(x) = \dfrac{1}{\sqrt[3]{x^3+1}}$, $\quad f'(x) = \dfrac{-x^2}{\left(\sqrt[3]{x^3+1}\right)^4}$, $\quad f''(x) = \dfrac{2x(x^3-1)}{\left(\sqrt[3]{x^3+1}\right)^7}$.

(e) On what intervals is $f(x)$ increasing? decreasing?

$$f'(x) > 0 \rightsquigarrow \frac{-x^2}{\left(\sqrt[3]{x^3+1}\right)^4} > 0$$

always ≥ 0

$-x^2 > 0$ never true.

$f'(x) \leq 0$ for all x. (and undefined at $x=-1$)

$f'(x) = 0$ when $x = 0$.

f is decreasing on

$(-\infty,-1) \cup (-1,0) \cup (0,\infty)$. Nowhere increasing.

(f) On what intervals is $f(x)$ concave up? concave down?

$$f''(x) = \frac{2x(x^3-1)}{\left(\sqrt[3]{x^3+1}\right)^7}$$

$x^3-1=0$ when $x=1$

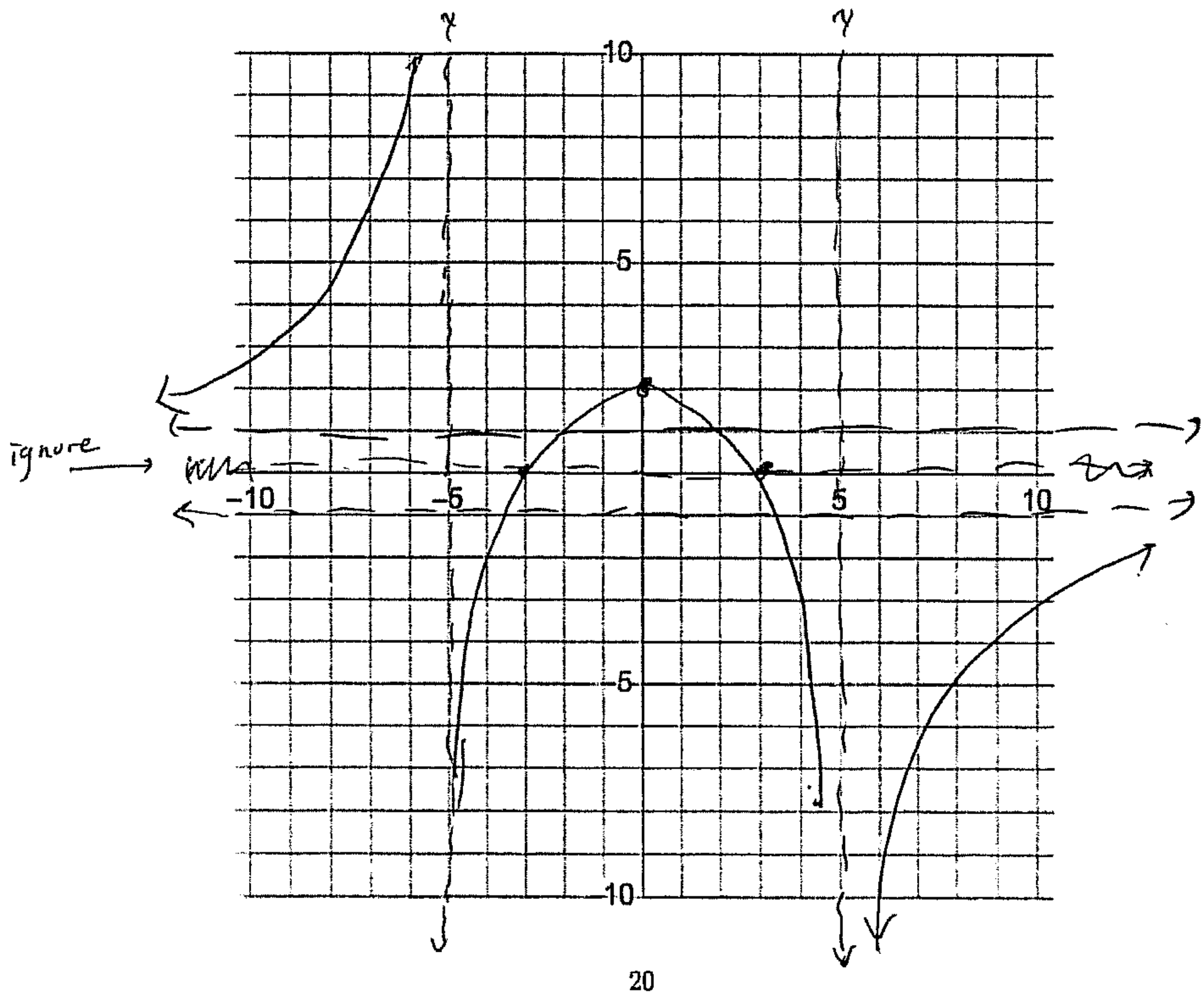
	-1		0		1	
x	$-$		$-$	$+$		$+$
x^3-1	$-$		$-$	$-$		$+$
$\left(\sqrt[3]{x^3+1}\right)^7$	$-$		$+$	$+$		$+$
$f''(x)$	$-$		$+$	$-$		$+$
	concave down		up	down		up

Concave up on $(-1,0) \cup (1,\infty)$

Concave down on $(-\infty,-1) \cup (0,1)$

13. (10 points) Sketch the graph of a function $f(x)$ that satisfies the following properties:

- x-intercepts: $-3, 3$
- y-intercepts: 2
- vertical asymptotes: $x = -5$ and $x = 5$
- horizontal asymptotes: $y = -1$ and $y = 1$
- $f'(x) > 0$ on $(-\infty, -5) \cup (-5, 0) \cup (5, \infty)$
- $f'(x) < 0$ on $(0, 5)$
- $f''(x) > 0$ on $(-\infty, -5)$
- $f''(x) < 0$ on $(-5, 5) \cup (5, \infty)$



14. (12 points)

(a) A particle is moving with the given velocity and position data. Find the position function $s(t)$ of the particle.

$$v(t) = 10\sin t + 3\cos t, \quad s(\pi/4) = 12$$

$$s(t) = \int v(t)\,dt = \int 10\sin t + 3\cos t\,dt = -10\cos t + 3\sin t + C$$

$$s\left(\frac{\pi}{4}\right) = -10\cos(\pi/4) + 3\sin(\pi/4) + C = 12$$

$$-10\cdot\frac{\sqrt{2}}{2} + 3\frac{\sqrt{2}}{2} + C = 12$$

$$C = 12 + \frac{7\sqrt{2}}{2}$$

$$\boxed{s(t) = -10\cos t + 3\sin t + 12 + \frac{7\sqrt{2}}{2}}$$

(b) Let $f(x) = \int_1^{x^2} (t-4)e^{-t^2}\,dt$ for all real numbers x. On what intervals is $f(x)$ an increasing function?

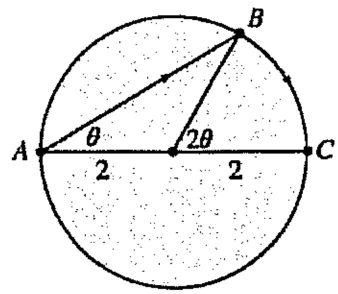
$$f'(x) = \frac{d}{dx}\left(\int_1^{x^2} (t-4)e^{-t^2}\,dt\right)$$

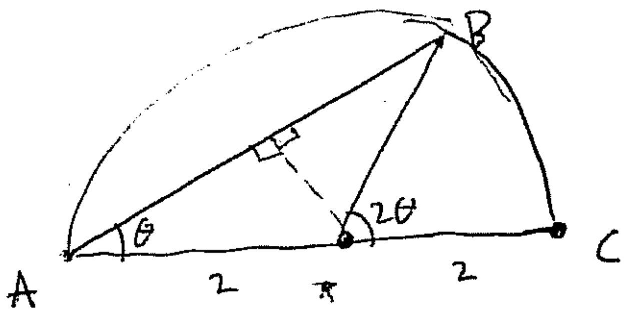
$$= 2x\,(x-4)e^{-x^2} = 0 \quad \text{when} \quad x = 0, 4$$

	$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$
x	$-$	$+$	$+$
$(x-4)$	$-$	$-$	$+$
$f'(x)$	$+$	$-$	$+$

$\boxed{\text{Increasing on } (-\infty, 0) \cup (4, \infty)}$

15. (15 points) A woman at a point A on the shore of a circular lake with radius 2 miles wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time (see the figure). She can walk at the rate of 4 miles/h and row a boat at 2 miles/h. How should she proceed? Justify your answer.





Length of arc $= \cancel{4}\ 2 \cdot 2\theta = 4\theta$.

Length from A to B $= 2 \cdot 2\cos\theta = 4\cos\theta$

$$T(\theta) = \frac{4\cos\theta}{2} + \frac{4\theta}{4} = 2\cos\theta + \theta \qquad 0 \le \theta \le \frac{\pi}{2}.$$

$$T'(\theta) = -2\sin\theta + 1 = 0$$

Critical point: $\theta = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

$$T(\pi/6) = 2\cos(\pi/6) + \frac{\pi}{6} = \sqrt{3} + \frac{\pi}{6}$$

$$T(0) = 2$$

$$T(\pi/2) = 2\cos\left(\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi/2$$ ← smallest value.

$\theta = \frac{\pi}{2}$ means she should walk the whole way.

Formula Sheet

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$$\begin{aligned}
\sin(x+y) &= \sin x\cos y + \cos x\sin y\\
\sin(x-y) &= \sin x\cos y - \cos x\sin y\\
\cos(x+y) &= \cos x\cos y - \sin x\sin y\\
\cos(x-y) &= \cos x\cos y + \sin x\sin y
\end{aligned}$$

$$\begin{aligned}
\sin 2x &= 2\sin x\cos x\\
\cos 2x &= \cos^2 x - \sin^2 x\\
&= 2\cos^2 x - 1\\
&= 1 - 2\sin^2 x
\end{aligned}$$

$$\begin{aligned}
\cos^2 x &= \frac{1+\cos 2x}{2}\\
\sin^2 x &= \frac{1-\cos 2x}{2}
\end{aligned}$$

Scratch Paper. You may tear this page off. Nothing you write on this page will be graded.