MATH 161
Final Exam
December 15, 2015

NAME (please print legibly): Solutions
Your University ID Number: ______

Bobkova (TR 9:40) Bridy (MW 2:00) Doyle (MWF 10:25)

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Please read the following instructions very carefully:

- By taking this exam, you are acknowledging that the following is prohibited by the College's Honesty Policy: obtaining an examination prior to its administration; using unauthorized aid during an examination or having such aid visible to you during an examination; knowingly assisting someone else during an examination or not keeping your work adequately protected from copying by another.
- Only pens/pencils and a single 3 in. × 5 in. index card with formulas are allowed. The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.
- Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given. Clearly circle or label your final answers.

Part A

QUESTION	VALUE	SCORE
1	10	
2	20	
3	10	
4	12	
5	15	
6	18	
7	15	
TOTAL	100	

Part B

QUESTION	VALUE	SCORE
8	12	
9	12	
10	18	
11	8	
12	15	
13	20	
14	15	
TOTAL	100	

Part A

1. (10 points) Mark the following statements as true (T) or false (F) by clearly circling the correct response. You do not need to explain your answers. No partial credit will be given.

(a) $\lim_{x \to 4^+} \frac{x}{(x-4)^2} = 0$

T F

(b) $\lim_{x \to -\infty} \frac{2x+4}{|x+2|} = 2.$

T F

(c) $\cos^{-1}(\cos(x)) = x$ for all $-\infty < x < \infty$

T F

(d) $\ln(x) - \ln(x^2) = \ln(1/x)$

T F

(e) $\lim_{x \to \infty} \frac{x + \sqrt{4x^2 - x}}{6x} = 1$

T F

2. (20 points) Compute the derivative (with respect to x) of each of the following functions:

(a) $\sin(3x) + \sin^3(x) + 3\sin^{-1}(x)$

$$\frac{dy}{dx} = 3\cos(3x) + 3(\sin(x))^2 \cdot \cos x + \frac{3}{\sqrt{1-x^2}}$$

(b) $xe^x \ln(x)$

$$f'(x) = \cancel{\tfrac{d}{dx}(xe^x)\,\ln(x)}\ \frac{d}{dx}(xe^x)\ln(x)$$

$$= \frac{1}{x}(xe^x) + \frac{d}{dx}(xe^x)\cdot \ln(x)$$

$$= \frac{1}{x}(xe^x) + (e^x + e^x \cdot x)\ln(x)$$

$$= e^x + \ln(x)e^x + \ln(x)e^x x$$

(c) $\dfrac{e^x + e^{-x}}{x^3 + x^{-3}}$

$$f'(x) = \frac{(e^x - e^{-x})(x^3 + x^{-3}) - (3x^2 - 3x^{-4})(e^x + e^{-x})}{(x^3 + x^{-3})^2}$$

$$= \frac{e^x - e^{-x}}{x^3 + x^{-3}} - \frac{3(x^2 - x^{-4})(e^x + e^{-x})}{(x^3 + x^{-3})^2}$$

(d) $\cos(x^2+\sqrt{x})$

$$f'(x) = -\sin(x^2+\sqrt{x}) \cdot \frac{d}{dx}(x^2+\sqrt{x})$$

$$= -\sin(x^2+\sqrt{x}) \cdot (2x + \frac{1}{2\sqrt{x}})$$

$$= -2x\sin(x^2+\sqrt{x}) - \frac{1}{2\sqrt{x}}\sin(x^2+\sqrt{x})$$

(e) $(\tan(x))^x$

$$y = (\tan(x))^x$$

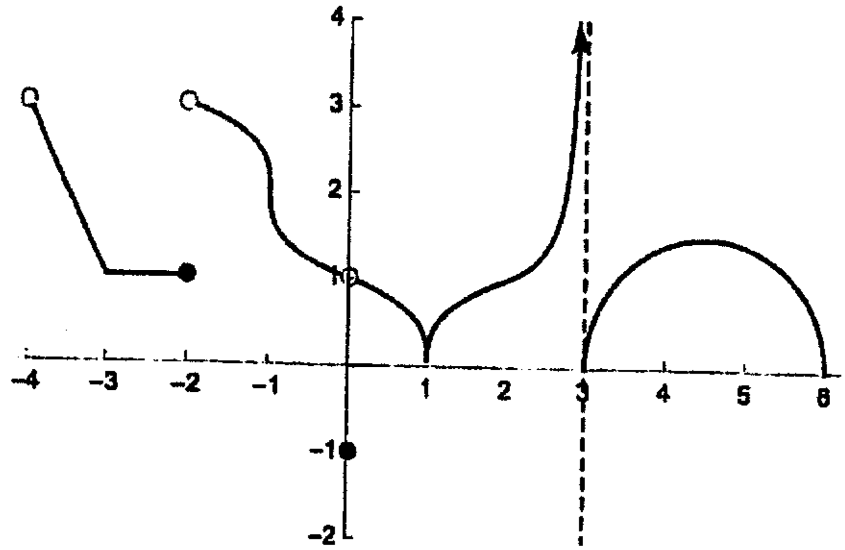
$$\ln y = x\ln(\tan x)$$

$$\frac{1}{y}\frac{dy}{dx} = \ln(\tan x) + x \cdot \frac{1}{\tan x} \cdot \sec^2 x$$

$$\frac{dy}{dx} = (\tan(x))^x \left(\ln(\tan x) + \frac{x \cdot \sec^2 x}{\tan x}\right)$$

$$= (\tan(x))^x \cdot \ln(\tan x) + (\tan(x))^{x-1} \cdot x \cdot \sec^2 x.$$

3. (10 points) Answer the following questions about the function $f(x)$, whose graph is shown below:



(a) For which values of a in the interval $(-4, 6)$ is $f(x)$ not continuous at a?

$a = -2, 0, -3$

(b) For which values of a in the interval $(-4, 6)$ is $f(x)$ not differentiable at a?

$a = -3, -2, -1, 0, 1, 3$

4. (12 points) Let k be a constant, and consider the function

$$f(x) = \begin{cases} e^{1/(x-2)}, & \text{if } x < 2 \\ 2x - k, & \text{if } x \geq 2. \end{cases}$$

(a) Compute $\lim_{x \to 2^-} f(x)$:

$\lim_{x \to 2^-} f(x) = e^{\frac{1}{x-2}}$ $\quad \because \lim_{x \to 2^-} \frac{1}{x-2} = -\frac{1}{0} = -\infty$

$= e^{-\infty}$

$= 0$

4

(b) Compute $\lim_{x \to 2^+} f(x)$:

$\lim_{x \to 2^+} f(x) = 2x - k$

$= 4 - k$

4

(c) For what value of k is this function continuous everywhere?

to be continuous : $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ $= f(x) = 0$

$\therefore \quad 4 - k = 0$

$k = 4$

4

15

5. (15 points) Consider the curve $x^2 + y^2 = y\cos x$.

(a) Compute $\frac{dy}{dx}$.

$$2x + 2y\frac{dy}{dx} = \cos x\frac{dy}{dx} + y\cdot(-\sin x)$$

$$(\cos x - 2y)\frac{dy}{dx} = 2x + \sin x\cdot y$$

$$\frac{dy}{dx} = \frac{2x + y\sin x}{\cos x - 2y}$$

(b) Give an equation for the tangent line to the curve at the point $(0, 1)$.

let $x = 0$, $y = 1$

$$\frac{dy}{dx} = \frac{0 + 1\cdot\sin(0)}{\cos(0) - 2}$$

$y = mx + c$

$$= \frac{0}{-1} = 0$$

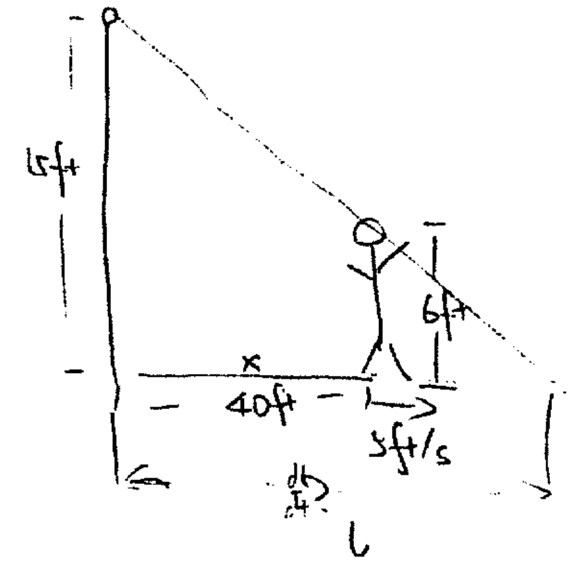
$$\therefore\ y - 1 = 0(x - 0)$$

$$y = 1$$

tangent line: $y = 1$

18

6. **(18 points)** A streetlight is mounted at the top of a 15-ft tall pole. A man 6 ft tall walks away from the pole with the speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft away from the pole?



let l be the length of tip of shadow to ~~str~~ pole

$\frac{dl}{dt}$ is ~~th~~ speed of tip of shadow.

let x be the distance of man to pole.

$$\frac{l-x}{l} = \frac{6ft}{15ft}$$

$$15l - 15x = 6l$$

$$9l = 15x$$

$$l = \frac{5}{3}x$$

$$\frac{dl}{dt} = \frac{5}{3}\frac{dx}{dt}$$

$$\frac{dl}{dt} = \frac{5}{3}\cdot 5 = \frac{25}{3}\ ft/s$$

$\therefore \frac{25}{3}$ ft/s

7. (15 points) Let $f(x) = \sin x$.

(a) Find the linearization of $f(x)$ at $x = \pi = 3.1415926\ldots$.

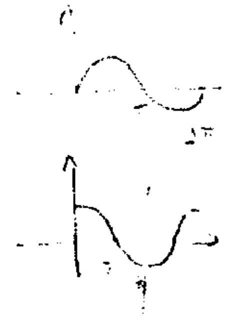
Linearization: $y - f(\pi) = f'(\pi)(x - \pi)$

$f(\pi) = \sin\pi = 0$

$f'(\pi) = \cos\pi = -1$ 5

$\therefore\ y - 0 = -(x - \pi)$

$y = -x + \pi$



(b) Use your answer to part (a) to approximate $\sin(3.1)$.

let $x = 3.1$

$y = -3.1 + \pi$

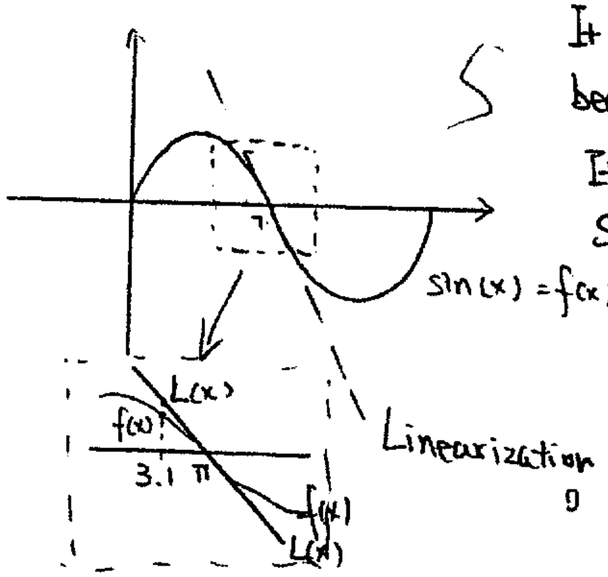
$= 3.1415926 - 3.1$

$= 0.0415926$

$\therefore\ \sin(3.1) = 0.0415926$

5

(c) Is your approximation in (b) an *over*estimate or an *under*estimate? Explain.



5

It is an overestimate.
because $f'(x)$ is decreasing in range $(0, \pi)$
It is con cave down.
So ~~sin~~ according to the graph.
$f(x) < L(x)$
so it is overestimate.

12

Part B

8. (12 points) Compute the following limits. Justify any use of L'Hospital's rule.

(a) $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$

$\cancel{\Rightarrow \lim_{x\to 1}\frac{x}{x-1} - \lim_{x\to 1}\frac{1}{\ln(x)}}$

$\cancel{\Rightarrow \infty}$

$\Rightarrow \lim_{x\to 1}\frac{x\ln x - x + 1}{(x-1)\ln(x)}$ $\frac{0}{0}$: use L'H rule

$$\frac{\ln(x)+x(\frac{1}{x})-1}{\ln(x)+\frac{1}{x}(x-1)} = \frac{\ln x}{\ln(x)+1-\frac{1}{x}} \quad \frac{0}{0} \text{ use L'H rule}$$

$$\Rightarrow \frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{x}{x+1} = \frac{1}{1+1} = \frac{1}{2}$$

(b) $\lim_{x\to 0^+} x^x$

$$\lim_{x\to 0^+} e^{x\ln(x)}$$

$\lim_{x\to 0^+} x\cdot\ln(x) \approx 0 \cdot -\infty$

$\therefore \lim_{x\to 0}\frac{\ln(x)}{\frac{1}{x}} \approx \frac{-\infty}{\infty}$ use LH rule

$$\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x$$

$\therefore = 0$

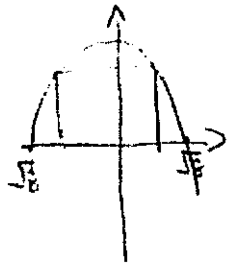
$\therefore \Rightarrow = e^0$

$= 1$

$\therefore \lim_{x\to 0^+} x^x = 1$

12

9. (12 points) Find the area of the largest rectangle which has two vertices on the x-axis and two vertices on the graph of the function $y = 8 - x^2$ with $-\sqrt{8} \le x \le \sqrt{8}$.



$$f(x) = 8 - x^2$$

$$A = 2x \cdot f(x)$$
$$= 2x(8 - x^2)$$
$$= 16x - 2x^3$$

$$A' = 16 - 6x^2 \quad \text{when } A' = 0$$

$$6x^2 = 16$$
$$x^2 = \frac{8}{3}$$
$$x = \frac{\pm\sqrt{8}}{\sqrt{3}}$$

A'	$\frac{\sqrt{8}}{2}$	$\frac{\sqrt{8}}{\sqrt{3}}$	$\sqrt{8}$
	$+$	0	$-$

$\therefore$ when $x = \frac{\sqrt{8}}{\sqrt{3}}$

$\therefore$ ~~A~~ $x = \frac{\sqrt{8}}{\sqrt{3}}$ A is maximised.

$$A_{max} = 16 \cdot \frac{2\sqrt{2}}{\sqrt{3}} - 2\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)^3$$
$$= \frac{32\sqrt{2}}{\sqrt{3}} - 2\left(\frac{8}{3} \cdot \frac{2\sqrt{2}}{\sqrt{3}}\right)$$
$$= \frac{32\sqrt{2}}{\sqrt{3}} - \frac{32}{3}\frac{\sqrt{2}}{\sqrt{3}}$$
$$= \frac{64}{3} \cdot \frac{\sqrt{2}}{\sqrt{3}}$$

10. (18 points) Consider the following function $f(x)$, with first and second derivative also given:

$$f(x) = \frac{\ln x}{x^2}, \quad f'(x) = \frac{1 - 2\ln x}{x^3}, \quad f''(x) = \frac{6\ln x - 5}{x^4}.$$

(a) Find the domain of $f(x)$.

Domain $= (0, \infty)$

(b) List all x- and y-intercepts of $f(x)$.

$x=0$ not in domain $\Rightarrow$ no y-int.

$\frac{\ln x}{x^2} = 0 \Rightarrow x = 1$ x-intercept

(c) List all vertical asymptotes of $f(x)$ or explain why none exist.

vertical asymptote at $x=0$ since

$$\lim_{x \to 0^+} \frac{\ln x}{x^2} = \lim_{x \to 0^+} \underbrace{\frac{1}{x^2}}_{\infty} \cdot \underbrace{\ln x}_{(-\infty)} = -\infty$$

Note: L'Hospital does not apply here since it is $\frac{\infty}{0}$.

(d) List all horizontal asymptotes of $f(x)$ or explain why none exist.

$$f(x) = \frac{\ln x}{x^2} \quad \text{when } x \to \infty$$

$$\lim_{x\to\infty} \frac{\ln x}{x^2} \quad \frac{\infty}{\infty} \quad \text{use LH}$$

$$\frac{\frac{1}{x}}{2x} \quad \text{LH}$$

$$\frac{1}{2x^2} = 0 \quad \therefore \ y = 0 \text{ when } x \to \infty$$

horizontal asymptote

$\because x > 0 \ \therefore$ no horizontal asymptotem for $x \to -\infty$

(e) On what intervals is $f(x)$ increasing? decreasing?

$$f'(x) = \frac{1-2\ln(x)}{x^3} \quad \text{when } f'(x) = 0$$

$$1 - 2\ln(x) = 0$$

$$\ln(x) = \frac{1}{2}$$

$$e^{\frac{1}{2}} = x$$

$(0, e^{\frac{1}{2}})$	$e^{\frac{1}{2}}$	$(e^{\frac{1}{2}}, \infty)$
+	0	−

$\therefore$ $f(x)$ increasing for $x \in (0, e^{\frac{1}{2}})$

$f(x)$ decreasing for $x \in (e^{\frac{1}{2}}, \infty)$

(f) On what intervals is $f(x)$ concave up? concave down?

$$f''(x) = \frac{6\ln(x) - 5}{x^4} \quad f''(x) = 0$$

$$6\ln(x) - 5 = 0$$

$$\ln(x) = \frac{5}{6}$$

$$e^{\frac{5}{6}} = x$$

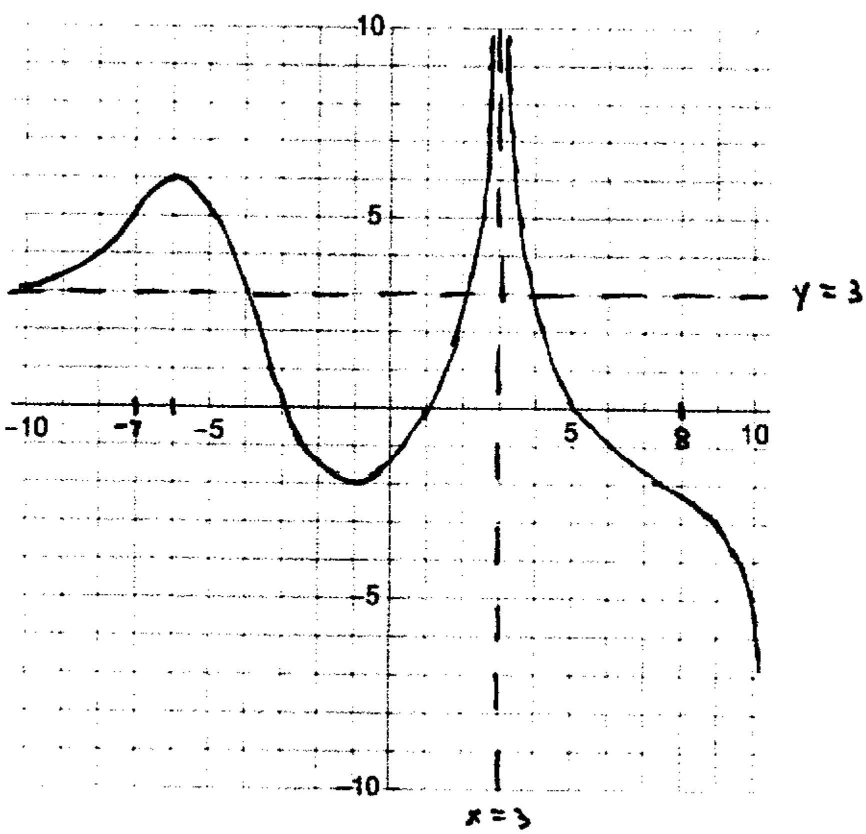
$(0, e^{\frac{5}{6}})$	$(e^{\frac{5}{6}}, \infty)$
−	+

$\therefore$ concave up for $x \in (e^{\frac{5}{6}}, \infty)$

Concave down for $x \in (0, e^{\frac{5}{6}})$

11. (8 points) Sketch the graph of a function $g(x)$ that satisfies the following properties:

- x-intercepts: $-3, 2, 5$
- y-intercept: -2
- vertical asymptote: $x = 3$
- horizontal asymptote: $y = 3$
- increasing on $(-\infty, -6) \cup (-1, 3)$
- decreasing on $(-6, -1) \cup (3, \infty)$
- concave up on $(-\infty, -7) \cup (-3, 3) \cup (3, 8)$
- concave down on $(-7, -3) \cup (8, \infty)$



12. (15 points)

(a) Find

$$\frac{d}{dx}\int_x^{e^x} \frac{t^2+1}{\sqrt{t+1}}\,dt$$

$$= \frac{d}{dx}\int_c^{e^x} \frac{t^2+1}{\sqrt{t+1}}\,dt - \frac{d}{dx}\int_c^{x} \frac{t^2+1}{\sqrt{t+1}}\,dt$$

$$= \frac{e^{2x}+1}{\sqrt{e^x+1}} \cdot e^x - \frac{x^2+1}{\sqrt{x+1}}$$

$$= \frac{e^{3x}+e^x}{\sqrt{e^x+1}} - \frac{x^2+1}{\sqrt{x+1}}$$

(b) Estimate the area under the graph $y=\sqrt{x}$ between $x=0$ and $x=8$ by using a Riemann sum with four intervals of equal width and right endpoints.

$\therefore$ interval $= \frac{8-0}{4} = 2 = \Delta x$

$x_i = a + 2i = 2i$

$\therefore$ Riemann sum : $\sum_{i=1}^{4} \Delta x \cdot f(x_i)$

$$= \sum_{i=1}^{4} 2 \cdot \sqrt{2i}$$

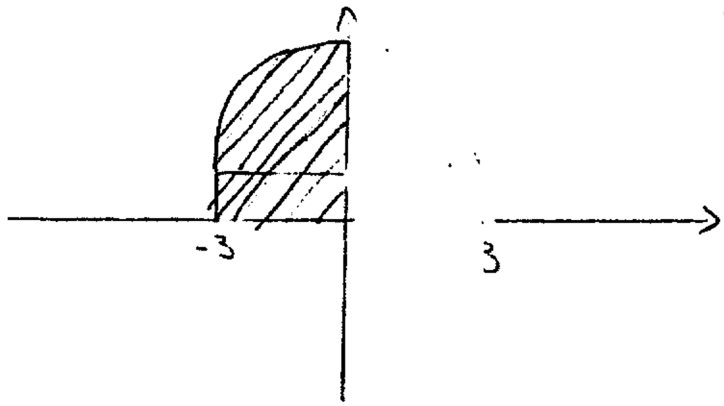
$$= 2\cdot\sqrt{2} + 2\cdot\sqrt{4} + 2\cdot\sqrt{6} + 2\cdot\sqrt{8}$$

$$= 2\sqrt{2} + 4 + 2\sqrt{6} + 4\sqrt{2}$$

$$= 6\sqrt{2} + 2\sqrt{6} + 4$$

(c) Compute the integral $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)\,dx$ by interpreting it as an area.

$\because\ r^2 = x^2 + y^2$



$$\int_{-3}^{0} (1 + \sqrt{9-x^2})\,dx$$

$$= \int_{-3}^{0} 1\,dx + \int_{-3}^{0} \sqrt{9-x^2}\,dx$$

$$= 1\cdot 3 + \pi\cdot 3^2\cdot\frac{1}{4}$$

Ø 5

$$A = 3 + \frac{9}{4}\pi$$

13. (20 points) Compute the following integrals.

(a) $\int \left(4x^3 - \frac{1}{\sqrt{x}} + \frac{1}{1+x^2}\right)\, dx$

$$= \int 4x^3\,dx - \int x^{-\frac{1}{2}}\,dx + \int \frac{1}{1+x^2}\,dx$$

$$= x^4 + C - \left(2\sqrt{x} + C\right) + \tan^{-1}(x) + C$$

$$= x^4 - 2\sqrt{x} + \tan^{-1}(x) + C$$

✓

(b) $\int_{-2\pi}^{-\pi} \left(e^{-2x} + \frac{1}{x} - \cos(x)\right)\, dx$

$$= \left(-\frac{1}{2}e^{-2x} + \ln(|x|) - \sin(x)\right)\Big|_{-2\pi}^{\pi}$$

$$= \left(\frac{1}{2}e^{2\pi} + \ln(\pi) + \sin(\pi)\right) - \left[-\frac{1}{2}e^{4\pi} + \ln(2\pi) + \sin(-2\pi)\right]$$

$$= -\frac{1}{2}e^{2\pi} + \ln(\pi) + 0 + \frac{1}{2}e^{4\pi} - \ln(2\pi) + 0$$

$$= \frac{1}{2}e^{4\pi} - \frac{1}{2}e^{2\pi} + \ln(\pi) - \ln(2\pi)$$

$$= \frac{1}{2}\left(e^{4\pi} - e^{2\pi}\right) + \ln\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\left(e^{4\pi} - e^{2\pi}\right) - \ln(2)$$

(c) $\int x^2\sqrt{x^3+1}\,dx$

$u = x^3+1$

$du = 3x^2dx \Rightarrow \frac{1}{3}du = x^2dx$

$$= \int \sqrt{u}\,\frac{1}{3}du = \frac{1}{3}\int u^{1/2}du$$

$$= \frac{1}{3}\,\frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{9}(x^3+1)^{3/2} + C$$

(d) $\int_0^{\pi/4} \cos^3(x)\sin(x)\;dx$

$u = \cos x$

$du = -\sin x\,dx$

$u(0) = \cos(0) = 1$

$u(\pi/4) = \cos(\pi/4) = \frac{\sqrt{2}}{2}$

$$= \int_{u=1}^{\sqrt{2}/2} -u^3du$$

$$= -\frac{1}{4}u^4\Big|_1^{\sqrt{2}/2}$$

$$= -\frac{1}{4}\left(\left(\frac{\sqrt{2}}{2}\right)^4 - 1^4\right)$$

$$= -\frac{1}{4}\left(\frac{1}{4} - 1\right) = \frac{3}{16}$$

14. **(15 points)** Aaron is driving his car along a straight path. After t seconds, the velocity of the car is given by $v(t) = 2t^2 - 10t + 8$ ft/s.

(a) Find the acceleration of the car at $t = 2$ seconds.

$a(t) = v'(t)$

$= 4t - 10$ ft/s^2

when $t = 2$

+ $a(t) = -2$

(b) Find the displacement of the car over the interval $0 \le t \le 3$.

$\int_0^3 v(t)\, dt$

$= \int_0^3 2t^2 - 10t + 8\, dt$

+ $= \frac{2}{3}t^3 - 5t^2 + 8t \Big|_0^3$

$= \frac{2}{3} \cdot 3^3 - 45 + 24 - 0$

$= 18 + 24 - 45 = -3$ ft

displacement $= -3$ ft

(c) Find the total distance traveled over the interval $0 \le t \le 3$.

when $v(t) = 0$

+ $2t^2 - 10t + 8 = 0$

$1 \quad -4$
$2 \quad -2$

$(t-4)(2t-2) = 0$

$t = 4, 1 \quad \because 0 \le t \le 3$

$\therefore t = 1$

	$(0,1)$	$(1,3)$
$v(t)$	+	−

$\therefore$ total distance : $\int_0^1 v(t) - \int_1^3 v(t)$

$= \frac{2}{3}t^3 - 5t^2 + 8t \Big|_0^1 - \left(\frac{2}{3}t^3 - 5t^2 + 8t\right)\Big|_1^3$

$= \left(\frac{2}{3} - 5 + 8\right) - \left(-3 - \frac{11}{3}\right)$

$= \frac{11}{3} + \frac{20}{3}$

$= \frac{31}{3}$ ft

$\therefore$ total distance $\frac{31}{3}$ ft