MATH 150 - WRITTEN HOMEWORK # 9

DUE FRIDAY, APRIL 19, 2024 AT 11:59 P.M.

Show your work clearly for each problem so that it can be understood how you arrived at your answer.

(1) *(8 points)* Consider a natural number n with a base 10 expansion as $111 \cdot \cdots \cdot 11$, where there are 3^k 1s in the base expansion. Prove using induction that n is divisible by 3^k.

Solution: We apply induction on k.

Base step: If $k = 1$, then we have $n = 111 = 3 \cdot 37$, thus n is divisible by 3.

Induction step: Assume that for some $k \in \mathbb{N}$, $n = 111 \cdot \cdots \cdot 11$ with 3^k 1s in the base 10 expansion is divisible by 3^k, i.e., $n = 3^k a$ for $a \in \mathbb{Z}$.

We want to show that $m = 111 \cdot \cdots \cdot 111$ with 3^{k+1} 1s in the base 10 expansion is divisible by 3^{k+1}.

Observe that m has 3 times as many 1s as in the expansion for n, and so we can write

$$m = n + 10^{3^k} n + 10^{2 \cdot 3^k} n = (1 + 10^{3^k} + 10^{2\cdot 3^k})n.$$

We can see that base 10 expansion of $1 + 10^{3^k} + 10^{2\cdot 3^k}$ contains 3 1s, thus it is divisible by 3 (since the sum of base 10 digits is 3), and we can write

$$1 + 10^{3^k} + 10^{2\cdot 3^k} = 3b$$

for some $b \in \mathbb{Z}$. Now, combining this together with the inductive hypothesis, we get

$$m = (1 + 10^{3^k} + 10^{2\cdot 3^k})n = (3b)(3^k a) = 3^{k+1}ab.$$

Hence, m is divisible by 3^{k+1}, as desired.

(2) *(8 points)* Consider the function recursively defined by

$$f(1) = 3, \quad f(2) = 2, \quad f(3) = 1,$$

and

$$f(n+1) = f(n) + f(n-1)f(n-2) \quad \text{for } \ n \geq 3.$$

Prove using strong induction that $f(n) \leq 2^{2^n}$ for all integers $n \geq 1$.

Solution: Let P(n): $f(n) \leq 2^{2^n}$.

Base step: If $n = 1$, then $f(1) = 3 \leq 2^2 = 4$, if $n = 2$, then $f(2) = 2 \leq 2^4$, and if $n = 3$, then $f(3) = 1 \leq 2^8$. Thus, P(1), P(2), and P(3) are true.

Induction step: Assume P(j) is true, i.e., $f(j) \le 2^{2^j}, \forall j \in \mathbb{Z}$ with $1 \le j \le k$, where $k \ge 3$.

We want to show that P($k+1$) is true, i.e., $f(k+1) \le 2^{2^{k+1}}$.

Observe that by recursive definition, we have

$$\begin{aligned} f(k+1) &= f(k) + f(k-1)f(k-2) \\ &\le 2^{2^k} + 2^{2^{k-1}}2^{2^{k-2}} \qquad \text{(by inductive hypothesis)} \\ &= 2^{2^k} + 2^{2^{k-1}+2^{k-2}} \\ &\le 2^{2^k} + 2^{2^k} \qquad \text{since } \ 2^{k-1} + 2^{k-2} \le 2 \cdot 2^{k-1} = 2^k \\ &= 2^{2^k+1}. \end{aligned}$$

Now observe that

$$2^k + 1 \le 2^k + 2^k = 2 \cdot 2^k = 2^{k+1},$$

whenever $k \ge 0$. Therefore, we conclude that

$$f(k+1) \le 2^{2^k+1} \le 2^{2^{k+1}},$$

completing the proof.

(3) *(8 points)* Give a proof by induction that for every positive integer n,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1).$$

Solution: *Base step:* If $n = 1$, then we have $2(\sqrt{2} - 1) = \sqrt{8} - 2 < 3 - 2 = 1$.

Induction step: Assume that for $n = k \ge 1$, we have

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} > 2(\sqrt{k+1} - 1).$$

We want to prove that for $n = k+1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - 1).$$

By induction hypothesis, we have

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}}.$$

Thus, we need to show that

$$2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - 1).$$

This is equivalent to prove that

$$\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - \sqrt{k+1}) = \frac{2}{\sqrt{k+2} + \sqrt{k+1}}.$$

Since

$$\frac{2}{\sqrt{k+2} + \sqrt{k+1}} \le \frac{2}{2\sqrt{k+1}} = \frac{1}{\sqrt{k+1}},$$

we get the desired conclusion.

(4) *(8 points)* Use induction to prove that for any positive integer n,

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}.$$

Solution: *Base step:* If $n = 1$, then we have

$$\sum_{i=1}^{1} \frac{1}{i^2} = 1 \leq 2 - \frac{1}{1} = 1,$$

which is clearly true.

Induction step: Assume that for $n = k \geq 1$, we have

$$\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}.$$

We want to show that for $n = k + 1$,

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}.$$

We use induction hypothesis to see that

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

Thus, we need to show that

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}.$$

Observe that $k^2 + 2k \leq k^2 + 2k + 1 \iff k(k+1) + k \leq (k+1)^2$. Dividing both sides by $k(k+1)^2$, we get

$$\frac{1}{k+1} + \frac{1}{(k+1)^2} \leq \frac{1}{k} \iff -\frac{1}{k} + \frac{1}{(k+1)^2} \leq -\frac{1}{k+1}.$$

Adding 2 on both sides, we obtain

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1},$$

completing the proof.

(5) *(8 points)* Prove using induction that $n^3 + n > 5n^2 + 4n - 8$ for all positive integers $n \geq 6$.

Solution: *Base step:* If $n = 6$, then we have $n^3 + n = 216 + 6 = 222$, while $5n^2 + 4n - 8 = 180 + 24 - 8 = 196$. Thus, $n^3 + n > 5n^2 + 4n - 8$ for $n = 1$.

Induction step: Assume that $n = k \geq 6$ and that $k^3 + k > 5k^2 + 4k - 8$.

We want to show that for $n = k+1$, we have that $(k+1)^3 + (k+1) > 5(k+1)^2 + 4(k+1) - 8$.

We see that

$$\begin{aligned}(k+1)^3+(k+1) &= (k^3+3k^2+3k+1)+(k+1)\\ &> (5k^2+4k-8)+(3k^2+3k+1) \qquad \text{(by induction hypothesis)}\\ &= 8k^2+7k-7.\end{aligned}$$

On the other hand,

$$5(k+1)^2+4(k+1)-8 = 5k^2+14k+1.$$

Thus, we need to show that

$$8k^2+7k-7 > 5k^2+14k+1 \iff 3k^2-7k-8>0.$$

Observe that the (real) quadratic equation $3x^2-7x-8=0$ has two real roots:

$$x=\frac{7\pm\sqrt{145}}{6}\approx -\frac{5}{6} \text{ and } \frac{19}{6}$$

since $\sqrt{145}\approx 12$. And, for $k \geq 6 > 4 >$ both of these roots, thus $3k^2-7k-8>0$, completing the proof.