MATH 150 - WRITTEN HOMEWORK # 7

DUE TUESDAY, MARCH 26, 2024 AT 11:59 P.M.

(1) *(8 points)* Prove that the sum of cubes of three consecutive integers is always divisible by 9.

Solution: For $n \in \mathbb{Z}$, consider $n-1)^3$, n^3, and $(n+1)^3$. Then

$$(n-1)^3 + n^3 + (n+1)^3 = 3n(n^2+2).$$

We either have that n is divisible by 3, then $3n$ is divisible by 9, so we are done. Or otherwise, we have $n = 3k + r$, where $r = 1, 2$ and $k \in \mathbb{Z}$. If $n = 3k+1$, then $n^2 + 2 = 9k^2 + 6k + 3 = 3(3k^2 + 2k + 1)$, which means that n^2+2 is divisible by 3, and thus, $3(n^2+2)$ is divisible by 9. If $n = 3k+2$, then $n^2 + 2 = 9k^2 + 12k + 6 = 3(3k^2 + 4k + 2)$, meaning that again n^2+2 is divisible by 3, and so in this case also $3(n^2+2)$ is divisible by 9. Either way, the product $3n(n^2+2)$ is divisible by 9.

(2) *(8 points)* Suppose that n and b are positive integers and $b \geq 2$. Prove that the base b representation of n has

$$\lfloor \log_b n \rfloor + 1$$

digits.

Solution: Let k be the number of digits in the base b representation of n. Observe that the smallest base b number with $k+1$ digits is b^k, and the largest base b number with k digits is $b^k - 1$. Thus, if $b^{k-1} \leq n < b^k$, then the base b representation of n has k digits. So for every n such that $b^{k-1} \leq n < b^k$, we have $k - 1 \leq \log_b n < k$. Since there is exactly one integer in a half-open interval of length one, we obtain $\lfloor \log_b n \rfloor = k - 1$. Hence, $k = \lfloor \log_b n \rfloor + 1$.

(3) *(8 points)* Use modular exponentiation to find $3^{(111)_{16}}$ **mod** 7, showing all of the steps in your work.

Solution: Write

$$(111)_{16} = 16^2 + 16 + 1 = (2^4)^2 + 2^4 + 2^0 = 2^8 + 2^4 + 2^0.$$

Computing

$$
\begin{aligned}
3^{2^0} \textbf{ mod } 7 &= 3 \textbf{ mod } 7 = 3 \quad \longleftarrow \\
3^{2^1} \textbf{ mod } 7 &= 9 \textbf{ mod } 7 = 2 \\
3^{2^2} \textbf{ mod } 7 &= 4 \\
3^{2^3} \textbf{ mod } 7 &= 16 \textbf{ mod } 7 = 2 \\
3^{2^4} \textbf{ mod } 7 &= 4 \quad \longleftarrow \\
3^{2^5} \textbf{ mod } 7 &= 16 \textbf{ mod } 7 = 2 \\
3^{2^6} \textbf{ mod } 7 &= 4 \\
3^{2^7} \textbf{ mod } 7 &= 16 \textbf{ mod } 7 = 2 \\
3^{2^8} \textbf{ mod } 7 &= 4 \quad \longleftarrow
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
3^{(111)_{16}} \textbf{ mod } 7 = 3^{2^8+2^4+2^0} \textbf{ mod } 7 &= (3^{2^8}\, 3^{2^4}\, 3^{2^0}) \textbf{ mod } 7 \\
&= (4 \cdot 4 \cdot 3) \textbf{ mod } 7 \\
&= (16 \cdot 3) \textbf{ mod } 7 = (2 \cdot 3) \textbf{ mod } 7 = 6.
\end{aligned}
$$

(4) (a) *(8 points)* Find $\gcd(74, 383)$ using the Euclidean Algorithm, showing all of your steps.

Solution:

$$
\begin{aligned}
383 &= 74(5) + 13 \\
74 &= 13(5) + 9 \\
13 &= 9(1) + 4 \\
9 &= 4(2) + 1 \\
4 &= 1(4) + 0.
\end{aligned}
$$

Thus, $\gcd(74, 383) = 1$.

(b) *(8 points)* Write $\gcd(74, 383)$ as a linear combination of 74 and 383 with integer coefficients. Show your work.

Solution: From part (a), we know that

$$
\begin{aligned}
13 &= 383 - 74(5) \\
9 &= 74 - 13(5) \\
4 &= 13 - 9(1) \\
1 &= 9 - 4(2) = 9 - (13 - 9(1))(2) \\
&= 9(3) - 13(2) = (74 - 13(5))(3) - 13(2) \\
&= 74(3) - 13(17) = 74(3) - (383 - 74(5))(17) \\
&= 74(88) + 383(-17).
\end{aligned}
$$