MATH 150 - WRITTEN HOMEWORK # 6

(1) (a) *(3 points)* Show that $x^3 - 6x^2 + 12x - 8$ is big - $\mathcal{O}$ of x^3. Please state the values used for witnesses C and k.

Solution: Observe that when $x \geq 1$, we have $1 \leq x \leq x^2 \leq x^3$. Therefore, we get

$$x^3 - 6x^2 + 12x - 8 \leq x^3 + 6x^2 + 12x + 8 \leq x^3 + 6x^3 + 12x^3 + 8x^3 = 27x^3.$$

Hence, $x^3 - 6x^2 + 12x - 8$ is big - $\mathcal{O}$ of x^3 with witnesses $k = 1$ andn $C = 27$.

(b) *(6 points)* Determine whether or not the statement

$$x \text{ is big - } \Omega \text{ of } x \ln x$$

is true. Prove your assertion.

Solution: If x is big - Ω of $x \ln x$, then there exists positive constants C, k such that

$$x \geq C\, x \ln x, \quad \forall x > k \iff \frac{1}{C} \geq \ln x, \quad \forall x > k \iff e^{1/C} \geq x, \quad \forall x > k.$$

However, C is a constant, thus if $x > \max\{k, e^{1/C}\}$, then $x > e^{1/C}$, which is a contradiction. Hence, x is not big - Ω of $x \ln x$.

(2) (a) *(6 points)* Determine whether or not the statement

$$7x^3 \ln x + 3x^2 + 22 \text{ is big - } \mathcal{O} \text{ of } x^3$$

is true. Prove your assertion.

Solution: If $7x^3 \ln x + 3x^2 + 22$ is big - $\mathcal{O}$ of x^3, then there exists positive constants C, k such that

$$7x^3 \ln x + 3x^2 + 22 \leq C\, x^3, \quad \forall x > k.$$

Since $3x^2 + 22 > 0$, a fortiori, we have

$$7x^3 \ln x \leq C\, x^3, \quad \forall x > k \iff \ln x \leq \frac{C}{7}, \quad \forall x > k.$$

Thus if $x > \max\{k, e^{C/7}\}$ (note that C is a constant), then $\ln x > C/7$ (i.e., $x > e^{C/7}$), which is a contradiction. Hence, $7x^3 \ln x + 3x^2 + 22$ is not big - $\mathcal{O}$ of x^3.

(b) *(3 points)* Is x^3 big - Θ of $7x^3 \ln x + 3x^2 + 22$? Prove or disprove.

Solution: No. For x^3 to be big - Θ of $7x^3 \ln x + 3x^2 + 22$, we need that x^3 is both big - $\mathcal{O}$ and big - Ω of $7x^3 \ln x + 3x^2 + 22$. However, we know from part (a) that

$$7x^3 \ln x + 3x^2 + 22 \text{ is not big - } \mathcal{O} \text{ of } x^3.$$

This is equivalent to the statement that

$$x^3 \text{ is not big - } \Omega \text{ of } 7x^3 \ln x + 3x^2 + 22.$$

Therefore, the second statement is false.

(3) (a) *(5 points)* Show that $\frac{x^2+1}{x+1}$ is big - $\mathcal{O}$ of x. Please state the values used for witnesses C and k.

Solution: We simplify the fraction first as follows:

$$\frac{x^2+1}{x+1} = \frac{x^2-1+2}{x+1} = \frac{x^2-1}{x+1} + \frac{2}{x+1} = x - 1 + \frac{2}{x+1}.$$

Thus, for $x > 1$, we have

$$\frac{x^2+1}{x+1} = x - 1 + \frac{2}{x+1} \leq x.$$

Hence, $\frac{x^2+1}{x+1}$ is big - $\mathcal{O}$ of x with witnesses $C = 1$ and $k = 1$.

(b) *(6 points)* Find the least integer n such that $\frac{x^4+x^2+1}{x^3+1}$ is big - $\mathcal{O}$ of x^n. Show why your answer works and state the values used for witnesses C and k.

Solution: Observe that $x^3 + 1 = (x+1)(x^2 - x + 1)$, we add and subtract x to the numerator and rearrange to again simplify the fraction first as follows:

$$\begin{aligned}\frac{x^4+x^2+1}{x^3+1} = \frac{(x^4+x)+(x^2-x+1)}{x^3+1} &= \frac{x(x^3+1)}{x^3+1} + \frac{x^2-x+1}{(x+1)(x^2-x+1)} \\ &= x + \frac{1}{x+1}.\end{aligned}$$

Thus, for $x > 1$, we have

$$\frac{x^4+x^2+1}{x^3+1} = x + \frac{1}{x+1} \leq x + x = 2x.$$

Hence, $\frac{x^4+x^2+1}{x^3+1}$ is big - $\mathcal{O}$ of x with witnesses $C = 2$ and $k = 1$. Therefore, $n = 1$.

(4) (a) *(5 points)* Show that $1 + 2 + 4 + 8 + \ldots + 2^n$ is big - Θ of 2^n. Please state the values used for constants C_1 and C_2 (that is, state what witnesses you use).

Solution: We need to prove that there are real numbers C_1 and C_2 and a positive real number k such that

$$C_1|2^n| \leq |1 + 2 + 4 + 8 + \ldots + 2^n| \leq C_2|2^n|$$

whenever $n > k$. The given series is a geometric series and since all the terms are positive, we have

$$2^n \leq 1 + 2 + 4 + 8 + \ldots + 2^n = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1 \leq 2^{n+1} = 2 \cdot 2^n.$$

Hence, $1 + 2 + 4 + 8 + \ldots + 2^n$ is big - Θ of 2^n with witnesses $C_1 = 1$ and $C_2 = 2$.

(b) *(6 points)* Show that $\lfloor x + \frac{3}{4} \rfloor$ is big - Θ of x. Please state the values used for constants k, C_1 and C_2 (that is, state what witnesses you use).

Solution: By definition of the floor function, $\lfloor x + \frac{3}{4} \rfloor \leq x + \frac{3}{4}$. If $x > \frac{3}{4}$, then $\lfloor x + \frac{3}{4} \rfloor < 2x$. We also have that $x + \frac{3}{4} < \lfloor x + \frac{3}{4} \rfloor + 1$, which can be rewritten as $x - \frac{1}{4} < \lfloor x + \frac{3}{4} \rfloor$. Now if $x > 1$, then $x - \frac{x}{4} < x - \frac{1}{4}$, i.e., $\frac{3x}{4} < x - \frac{1}{4}$. Therefore, $\frac{3x}{4} < \lfloor x + \frac{3}{4} \rfloor$. Hence, for $x > 1$,

$$\frac{3x}{4} < \lfloor x + \frac{3}{4} \rfloor < 2x.$$

This shows that $\lfloor x + \frac{3}{4} \rfloor$ is big - Θ of x with the witnesses $C_1 = \frac{3}{4}$, $C_2 = 2$ and $k = 1$.