MATH 150 - WRITTEN HOMEWORK # 5 - SOLUTIONS

(1) *(10 points)*

(a) *(2 points)* There are four different functions $f : \{a, b\} \to \{0, 1\}$. List them all.

Solution: Four functions are: $f_1 = \{(a, 0), (b, 0)\}$, $f_2 = \{(a, 1), (b, 0)\}$, $f_3 = \{(a, 0), (b, 1)\}$, and $f_4 = \{(a, 1), (b, 1)\}$.

(b) *(5 points)* Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $f(n) = 2n + 3$. Determine if f is one-to-one, onto, bijective, or neither. Justify your answer!

Solution: This function is *one-to-one*. To see this, suppose $m, n \in \mathbb{Z}$ and $f(m) = f(n)$. This means that $2m+3 = 2n+3$, which implies that $2m = 2n$, and thus $m = n$. Therefore, f is one-to-one.
This function is *not surjective*. To see this, observe that $f(n)$ is odd for all $n \in \mathbb{Z}$. So given the (even) number, say 2 in the codomain $\mathbb{Z}$, there is no n with $f(n) = 2$.

(c) *(3 points)* Let $g : \mathbb{R} \to \mathbb{Z}$ be a function defined by $g(x) = \lfloor x \rfloor$. Determine if g is one-to-one, onto, bijective, or neither. Justify your answer!

Solution: This function is *onto*. To see this, observe that the range, and co-domain is, $\mathbb{Z}$, the set of all integers. Also for any integer, n, $g(n) = \lfloor n \rfloor = n$, meaning that every integer, n, is reached, or obtained by the floor function, $g(x) = \lfloor x \rfloor$.
This function is *not one-to-one*. To see this, observe that multiple numbers are rounded down to the same integer. For example, $g(3.4) = \lfloor 3.4 \rfloor = 3$, and $g(3.7) = \lfloor 3.7 \rfloor = 3$, so that $g(3.4) = g(3.7)$, but $3.4 \neq 3.7$.

(2) *(12 points.)*

(a) Let $g : A \to B$ and $f : B \to C$ be two functions. Show that if g and f are both injective, then $f \circ g : A \to C$ is injective.

Solution: To show that $f \circ g : A \to C$ is injective we must show that for all $a, b \in A$, $f(g(a)) = f(g(b)) \implies a = b$. Consider the function f, we know that f is injective (that is, $f(a) = f(b) \implies a = b$. Then from this we deduce that the statement $f(g(a)) = f(g(b)) \implies g(a) = g(b)$, but g is also injective which implies that $a = b$. Therefore, $f \circ g : A \to C$ is injective.

(b) Let $f : A \to B$ and $g : B \to C$ be two functions. Show that if $g \circ f : A \to C$ is surjective, then g is surjective.

Solution: Assume that $g \circ f : A \to C$ is surjective and then consider the case where g is not surjective. Now $g \circ f : A \to C$ is surjective means that for all $c \in C$, there exists $a \in A$ such that $g(f(a)) = c$, where $f(a) \in B$. However, if g is not surjective, then this implies that there exists $c \in C$ such that there is no $f(a) \in B$ such that $g(f(a)) = c$, which contradicts the surjectivity of $g \circ f : A \to C$. Hence, if $g \circ f : A \to C$ is surjective, then g must also be surjective.

Alternatively, suppose that $g \circ f : A \to C$ is surjective. Let $c \in C$. Since $g \circ f$ is surjective, there exists $a \in A$ such that $g(f(a)) = c$, where $f(a) \in B$. Therefore, if we let $b = f(a) \in B$, then we see that for every $c \in C$ there exists $b \in B$ such that $g(b) = c$. Hence, g is surjective.

(3) *(8 points.)* Prove that the function $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{8\}$ defined by $f(x) = \dfrac{8x+3}{x-3}$ is bijective.

Solution: Need to show that the function f is both injective and surjective.

Injective: Assume $x, y \in \mathbb{R} - \{3\}$ and

$$\frac{8x+3}{x-3} = \frac{8y+3}{y-3} \iff (8x+3)(y-3) = (8y+3)(x-3).$$

Expanding both sides leads to

$$-24x + 3y = -24y + 3x \iff 27y = 27x \iff y = x.$$

Thus, f is injective.

Surjective: For any $y \in \mathbb{R} - \{8\}$, take $x = \dfrac{3(y+1)}{y-8}$ and note that $x \neq 3$. Then

$$f(x) = \frac{8\left(\frac{3(y+1)}{y-8}\right) + 3}{\frac{3(y+1)}{y-8} - 3} = \frac{\frac{27y}{y-8}}{\frac{27}{y-8}} = y.$$

Thus, f is surjective. Hence, f is bijective.

(4) *(10 points.)* Prove that the function $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = \dfrac{(-1)^n(2n-1)+1}{4}$ is bijective.

Solution: To show f is injective, assume $m, n \in \mathbb{N}$ and $f(m) = f(n)$. This means that

$$\frac{(-1)^m(2m-1)+1}{4} = \frac{(-1)^n(2n-1)+1}{4} \iff (-1)^m(2m-1) = (-1)^n(2n-1).$$

We first show that m and n have same parity. Arguing by contradiction, assume m is even and n is odd. Then the previous expression reduces to $2m - 1 = -(2n-1)$, which implies $m + n = 1$. This contradicts the fact that $m, n \in \mathbb{N}$. So m and n must be both even or both odd. Now in either case, $(-1)^m(2m-1) = (-1)^n(2n-1)$ yields $2m-1 = 2n-1$, which implies $m = n$. Thus, f is injective.

To prove f is surjective, for any $m \in \mathbb{Z}$, take $n = 2m$, is $m > 0$, and $n = -2m+1$, if $m \leq 0$. Observe that $n \in \mathbb{N}$. Now when $m > 0$,

$$f(n) = \frac{(-1)^n(2n-1)+1}{4} = \frac{(-1)^{2m}(2(2m)-1)+1}{4} = m,$$

and when $m \leq 0$,

$$f(n) = \frac{(-1)^n(2n-1)+1}{4} = \frac{(-1)^{-2m+1}(2(-2m+1)-1)+1}{4} = m.$$

So f is surjective. Therefore, f is bijective.