MATH 150 - WRITTEN HOMEWORK # 4 - SOLUTIONS

(1) *(15 points.)* Let $A = \{0, 1, 4, 5, 6, 7, 9, 10, 11\}$, $B = \{x \mid x \text{ is an even integer and } 3 \leq x \leq 8\}$, and $C = \{a, b, c, 1, 2, 3\}$.

(a) Replace the blank with the most appropriate symbol ($\in, \notin, \subseteq, \not\subseteq$)

Solution:

$8 \in B, \quad 8 \notin A, \quad 7 \notin B, \quad 4 \in A, \quad c \in C, \quad a \notin B, \quad \{4, 5, 9\} \subseteq A,$
$\{4, 5, 8\} \not\subseteq B, \quad \{a, 2, c\} \subseteq C.$

(b) Compute: $A \cup B, \quad A \cap B, \quad A - B, \quad B - A, \quad \mathcal{P}(A \cap B), \quad (A \cap C) \times \{a, b, c\}$

Solution:

$A \cup B = \{0, 1, 4, 5, 6, 7, 8, 9, 10, 11\}, \quad A \cap B = \{4, 6\}, \quad A - B = \{0, 1, 5, 7, 9, 10, 11\},$
$B - A = \{8\}, \quad \mathcal{P}(A \cap B) = \{\emptyset, \{4\}, \{6\}, \{4, 6\}\},$
$(A \cap C) \times \{a, b, c\} = \{(4, a), (4, b), (4, c), (6, a), (6, b), (6, c)\}.$

(2) *(15 points.)* Let A, B and C be arbitrary sets. Prove or give a counterexample to the following statements:

(a) $A - (B \cap C) = (A - B) \cup (A - C)$.

Solution: We need to show that

(i) $A - (B \cap C) \subseteq (A - B) \cup (A - C)$, and
(ii) $(A - B) \cup (A - C) \subseteq A - (B \cap C)$.

To prove (i), let $x \in A - (B \cap C)$. Then $x \in A$ and $x \notin (B \cap C)$. It follows from the later that $x \notin B$ or $x \notin C$. If $x \in A$, then $x \in A - B$, and thus, $x \in (A - B) \cup (A - C)$. For the second case, if $x \in A$, then $x \in A - C$, which again implies that $x \in (A - B) \cup (A - C)$. Thus, we always have $x \in (A - B) \cup (A - C)$. Hence, $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.

To prove (ii), let $x \in (A - B) \cup (A - C)$. Then either $x \in A - B$ or $x \in A - C$. If $x \in A - B$, then $x \in A$ and $x \notin B$. Since $x \notin B$, we have that $x \notin B \cap C$. Thus, $x \in A$ and $x \notin B \cap C$, so $x \in A - (B \cap C)$. Now if $x \in A - C$, then $x \in A$ and $x \notin C$. Since $x \notin C$, we have that $x \notin B \cap C$. Thus, as before $x \in A$ and $x \notin B \cap C$, and we have that $x \in A - (B \cap C$. Hence, $(A - B) \cup (A - C) \subseteq A - (B \cap C)$.

(b) $(A - B) \cup C = (A \cup B \cup C) - (A \cap B)$.

Solution: Not true. Let A and C bet empty sets, but B not an empty set. Suppose $x \in B$. Then $A - B$ is empty, so $(A - B) \cup C$ is empty. On the other hand, $A \cap B$ is empty, and $x \in A \cup B \cup C$, so $x \in (A \cup B \cup C) - (A \cap B)$, which means that $(A \cup B \cup C) - (A \cap B)$ is not empty. Thus, not equal to $(A - B) \cup C$.

(3) *(10 points.)* Suppose A and B are sets. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Solution: Need to prove that

(a) $A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$, and

(b) $\mathcal{P}(A) \subseteq \mathcal{P}(B) \implies A \subseteq B$.

To prove (a), suppose $A \subseteq B$, then $(\forall x \in A)\, x \in B$. Thus, if $C \in \mathcal{P}(A)$, then $C \subseteq A$. But then, since $A \subseteq B$, it follows that $C \subseteq B$. This means that $C \in \mathcal{P}(B)$. Hence, $(\forall C \in \mathcal{P}(A))\, C \in \mathcal{P}(B)$, so therefore $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

To prove (b), suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then in particular for any $x \in A$, the singleton set $\{x\} \subseteq A$, so $\{x\} \in \mathcal{P}(A)$. But then, since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, it follows that $\{x\} \in \mathcal{P}(B)$. This means that $\{x\} \subseteq B$, thus $x \in B$. Since this holds for all $x \in A$, we have $A \subseteq B$.