MATH 150 - WRITTEN HOMEWORK # 3 - SOLUTIONS

(1) *(8 points.)* Let the domain $\mathbb{R} = (-\infty, \infty)$ consists of all real numbers. Determine the truth value of each of the following statements. If the statement is True, justify your answer. If the statement is False, give a counterexample.

(a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\,(0 < x - y < 3)$.

Solution: True. For any x, take $y = x - 1$. Then $x - y = 1$, which is strictly between 0 and 3.

(b) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})\,(x^2 = y^2 \to x = y)$.

Solution: False. Counterexample: Take $x = 1$ and $y = -1$. Then $x^2 = y^2$ but $x \neq y$.

(c) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\exists z \in \mathbb{R})\,((y \neq z) \wedge (x^2 = y^2) \wedge (x^2 = z^2))$.

Solution: False. Counterexample: Take $x = 0$. Then $\forall y\, \forall z$, $0 = x^2 = y^2$ and $0 = x^2 = z^2$ implies that $y = 0 = z$.

(d) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\,((x < y) \to (y^2 > 4))$.

Solution: True. Take $x = 2$. (One could pick x to be any real number greater than or equal to 2). Then for any $y \leq 2$, the hypothesis is false, thus the implication is vacuously true. On the other hand, for any $y > 2$, we have $y^2 > 4$ and the implication is again true.

(2) *(16 points.)*

(a) Let a and b be positive real numbers. Prove that if $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$.

Solution: Suppose $a \leq b$. Subtracting b from both sides gives $a - b \leq 0$, which can be written as $(\sqrt{a})^2 - (\sqrt{b})^2 \leq 0$. Factoring this as a difference of two squares, we have $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) \leq 0$. Dividing both sides by the positive real number $\sqrt{a} + \sqrt{b}$ gives $\sqrt{a} - \sqrt{b} < 0$. Adding $\sqrt{b}$ to both sides yields $\sqrt{a} \leq \sqrt{b}$, as desired.

(b) Prove that if a and b are positive real numbers, then $2\sqrt{ab} \leq a + b$.

Solution: Suppose a and b are positive real numbers. Observe that $0 \leq (a - b)^2$, that is, $0 \leq a^2 - 2ab + b^2$. Adding $4ab$ to both sides gives $4ab \leq a^2 + 2ab + b^2$. Factoring the expression on the right-hand side of the inequality yields $4ab \leq (a + b)^2$. By part (a), such an inequality still holds after taking the square root of both sides; thus, we obtain $2\sqrt{ab} \leq a + b$, as desired.

(3) *(7 points.)* Prove that there does not exist integers x and y such that $7x^2 + 2y^4 = 31$.

Solution: Observe that since the equation only depends on x^2, y^4, without loss of generality we can assume integers $x \geq 0, y \geq 0$. Assume for the sake of contradiction that (x, y) is a solution, then since x^2, $y^4 \geq 0$, we must have $7x^2 \leq 31$ and $2y^4 \leq 31$. Thus, $x = 0, 1,$ or 2, while $y = 0$, or 1. Searching through these 6 possibilities, we see that there are no pairs (x, y) satisfying the equation, a contradiction.

Alternative Proof: If (x, y) is a solution, then since $2y^4$ is even, $7x^2 = 31 - 2y^4$ is the difference of odd and even numbers and must be odd. If x were even, then so would x^2 and hence, also $7x^2$. Hence, x must be odd. Now as in the previous Proof, the only possible values of x are $0, 1$ or 2, and thus, $x = 1$. That would imply $7 \cdot 1^2 + 2y^4 = 31$, or $y = (12)^{1/4} \notin \mathbb{Z}$. Contradiction.

(4) *(9 points.)* Prove that for any integer n, the following statements are equivalent:

(a) $n^2 + 1$ is odd.

(b) $1 - n$ is odd.

(c) n^3 is even.

Solution: We start by defining the following propositions:

- $p \,:\, n^2 + 1$ is odd,
- $q \,:\, 1 - n$ is odd,
- $r \,:\, n^3$ is even.

We will show that (i) $q \leftrightarrow p$, and (ii) $q \leftrightarrow r$.

➤ **Proof of (i):** $q \leftrightarrow p$, i.e., $1 - n$ is odd $\leftrightarrow n^2 + 1$ is odd. To establish this, we need to show two implications: $q \to p$, i.e., $1 - n$ is odd implies $n^2 + 1$ is odd and $p \to q$, i.e., $n^2 + 1$ is odd implies $1 - n$ is odd.

➢ Proof of $q \to p$: Suppose $1 - n$ is odd. Then there exists an integer k such that $1 - n = 2k + 1$. Thus, $n = -2k$ and taking the square on both sides, we get $n^2 = 4k^2$. So, $n^2 + 1 = 4k^2 + 1 = 2(2k^2) + 1$. Since $2k^2$ is an integer because k is an integer, we see that $n^2 + 1$ is odd.

➢ Proof of $p \to q$: We prove this by showing a contrapositive, i.e., $\neg q \to \neg p$. Here,

★ $\neg q \,:\, 1 - n$ is even,

★ $\neg p \,:\, n^2 + 1$ is even.

Suppose $1 - n$ is even. Then there exists an integer k such that $1 - n = 2k$. Thus, $n = 1 - 2k$ and taking the square on both sides, we get $n^2 = (1-2k)^2 = 4k^2 - 4k + 1$. So, $n^2 + 1 = 4k^2 - 4k + 2 = 2(2k^2 - 2k + 1)$. Since $2k^2 - 2k + 1$ is an integer because k is an integer, we see that $n^2 + 1$ is even.

➤ **Proof of (ii):** $q \leftrightarrow r$, i.e., $1 - n$ is odd $\leftrightarrow n^3$ is even. To establish this, we need to show two implications: $q \to r$, i.e., $1 - n$ is odd implies n^3 is even and $r \to q$, i.e., n^3 is even implies $1 - n$ is odd.

➢ Proof of $q \to r$: Suppose $1 - n$ is odd. Then there exists an integer k such that $1 - n = 2k + 1$. Thus, $n = -2k$ and taking the cube on both sides, we get $n^3 = 8k^3 = 2(4k^3)$. Since $8k^3$ is an integer because k is an integer, we see that n^3 is even.

➢ Proof of $r \to q$: We prove this by showing a contrapositive, i.e., $\neg q \to \neg r$. Here,

★ $\neg q \,:\, 1 - n$ is even,

★ $\neg r \,:\, n^3$ is odd.

Suppose $1 - n$ is even. Then there exists an integer k such that $1 - n = 2k$. Thus, $n = 1 - 2k$ and taking the cube on both sides, we get $n^3 = (1 - 2k)^3 = -8k^3 + 12k^2 - 6k + 1 = 2(-4k^3 + 6k^2 - 3k) + 1$. Since $-4k^3 + 6k^2 - 3k$ is an integer because k is an integer, we see that n^3 is odd.