Math 150: Discrete Mathematics

Midterm 1

October 11, 2018

NAME (please print legibly): Solutions

Your University ID Number: ______

Indicate the lecture time you attend with a check in the appropriate box:

S. Amelotte	MW 3:25–4:40pm	
A. Iosevich	MW 10:25–11:40am	
J. Passant	MW 9:00–10:15am	
V. Petkov	MW 12:30–1:45pm	

- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 11 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Please sign the pledge below.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______

QUESTION	VALUE	SCORE
1	15	
2	15	
3	20	
4	15	
5	15	
6	20	
TOTAL	100	

1. (15 points)

Translate the following statements into a logical expressions using the standard set notation.

(a) "The sum of two integers is always an integer".

$$(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})[x+y \in \mathbb{Z}].$$

(b) "For every two positive integers, there exists another integer which when added to the first integer, gives the second".

$$(\forall x \in \mathbb{Z}^{+})(\forall y \in \mathbb{Z}^{+})(\exists z \in \mathbb{Z})[x+z=y].$$

(c) "There exist real numbers such that the sum of their squares is not an integer".

$$(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})[\neg((x^2+y^2) \in \mathbb{Z})]$$

equiv to $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})[x^2+y^2 \in \mathbb{R}-\mathbb{Z}].$

2. (15 points)

(a) Use the insertion sorting algorithm to order the following integers:

$$12, 4, 2, 11, 1, 6, 5, 9, 7.$$

Step 1 12

Step 2. $4 < 12$ so new list is 4, 12.

Step 3 $2 < 12$ so new list is 2, 4, 12.

Step 4 $11 \not< 2$, $11 \not< 4$, $11 < 12$ so new list is
2, 4, 11, 12.

Step 5. $1 < 2$ so new list is 1, 2, 4, 11, 12.

Step 6 $6 \not< 1$, $6 \not< 2$, $6 \not< 4$, $6 < 11$ so new list is
1, 2, 4, 6, 11, 12.

Step 7: $5 \not< 1$, $5 \not< 2$, $5 \not< 4$, $5 < 6$ so new list is
1, 2, 4, 5, 6, 11, 12.

Step 8 $9 \not< 1$, $9 \not< 2$, $9 \not< 4$, $9 \not< 5$, $9 \not< 6$, $9 < 11$
So new list is
1, 2, 4, 5, 6, 9, 11, 12.

Step 9: $7 \not< 1$, $7 \not< 2$, $7 \not< 4$, $7 \not< 5$, $7 \not< 6$, $7 < 9$
So new list is
1, 2, 4, 5, 6, 7, 9, 11, 12.

Step 10: No new numbers, so final list is
1, 2, 4, 5, 6, 7, 9, 11, 12.

(b) One defines the trinary search algorithm to find x in a list of sorted numbers $a_1, \ldots, a_n$ in the following way:

- Divide the list into three sections, a_1 through $a_{\lfloor \frac{n}{3} \rfloor}$, $a_{\lceil \frac{n}{3} \rceil}$ through $a_{\lfloor \frac{2n}{3} \rfloor}$ and, $a_{\lceil \frac{2n}{3} \rceil}$ through a_n.
- If $x < \lfloor \frac{n}{3} \rfloor$ choose the first third of the list, if $a_{\lceil \frac{n}{3} \rceil} \leq x \leq a_{\lfloor \frac{2n}{3} \rfloor}$ choose the second third and, if $x > a_{\lceil \frac{2n}{3} \rceil}$ choose the final third. This third is considered the new list.
- Repeat steps one and two with this new list, until the list has length one.

Use trinary search on your ordered list from part (a) to find the number 11.

List from part (a) is

$$\underbrace{1, 2, \overset{a_3}{4}}, \underbrace{5, 6, \overset{a_6}{7}}, \underbrace{9, 11, 12}.$$

n is 9 so $\lfloor \frac{n}{3} \rfloor = 3$. $\lfloor \frac{2n}{3} \rfloor = 6$.

As $11 > a_6$, we choose the third part of our list.

Our new list is $\overset{\tilde{a}_1}{9}, \overset{\tilde{a}_2}{11}, \overset{\tilde{a}_3}{12}$.

$n = 3$ So $\frac{n}{3} = 1$ $\frac{2n}{3} = 2$.

So part 1 is $\underbrace{9}$ part 2 is $\underbrace{11}$ part 3 is $\underbrace{12}$.

As $11 = \tilde{a}_2$ we choose the middle part of our list.

This list is of length one, so we are done.

$11 = \tilde{a}_2 = a_7$.

3. (20 points)

Prove the following identities.

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

$\subseteq$ Let $x \in A \cup (B \cap C)$

If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$

So $x \in (A \cup B) \cap (A \cup C)$.

If $x \in B \cap C$ then $x \in B$ and $x \in C$

so $x \in B \cup A$ and $x \in C \cup A$

~~So $x \in (A \cup B) \cap (C \cup A)$~~ So, $x \in (A \cup B) \cap (A \cup C)$.

$\supseteq$ If $x \in (A \cup B) \cap (A \cup C)$ Then, $x \in A \cup B$ and $x \in A \cup C$.

If $x \in A$ then $x \in A \cup (B \cap C)$ and done, so assume $x \notin A$.

Then $x \in A \cup B \wedge x \notin A \Rightarrow x \in B$. $x \in A \cup C \wedge x \notin A \Rightarrow x \in C$ $\} \Rightarrow x \in B \cap C$.

So $x \in A \cup (B \cap C)$.

(b) $A \cup (A \cap B) = A$.

$\subseteq$ Let $x \in A \cup (A \cap B)$. then $x \in A$ or $x \in A \cap B$.

If $x \in A$ then done, else $x \in A \cap B \Rightarrow x \in A$.

$\supseteq$ If $x \in A$ then $x \in A \cup (A \cap B)$.

(c) $\overline{(A \cap B) \cap C} = (\overline{A} \cup \overline{B}) \cup \overline{C}$.

If $x \in \overline{(A \cap B) \cap C} \Leftrightarrow x \notin (A \cap B) \cap C$

ie $\Leftrightarrow$ $\cancel{x \notin}$ $\neg (x \in (A \cap B) \cap C)$

$\Leftrightarrow \neg ((x \in A \wedge x \in B) \wedge x \in C)$

$\Leftrightarrow (x \notin A) \vee (x \notin B) \vee (x \notin C)$

$\Leftrightarrow (x \in \overline{A}) \vee (x \in \overline{B}) \vee (x \in \overline{C})$

$\Leftrightarrow x \in (\overline{A} \cup \overline{B}) \cup \overline{C}$.

As $\Leftrightarrow$ at ~~both~~ all stages we are done. //

4. (15 points)

Prove that the following are logically equivalent using truth tables.

(a) $\neg(\neg p) \equiv p$.

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

So $p \leftrightarrow \neg(\neg p)$ is a tautology.

(b) $\neg(p \to q) \equiv p \wedge \neg q$.

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$p \wedge \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

So $\neg(p \to q) \leftrightarrow p \wedge \neg q$ is a tautology.

(c) $(p_1 \vee p_2 \vee p_3) \to q \equiv (p_1 \to q) \wedge (p_2 \to q) \wedge (p_3 \to q)$.

I will show $(p_1 \vee p_2) \to q \equiv (p_1 \to q) \wedge (p_2 \to q)$.
The full question is too long.

p_1	p_2	q	$p_1 \to q$	$p_2 \to q$	$(p_1 \to q) \wedge (p_2 \to q)$	$(p_1 \vee p_2) \to q$
T	T	T	T	T	T	T
T	T	F	F	F	F	F
T	F	T	T	T	T	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
F	F	T	T	T	T	T
F	T	F	T	T	F	F
F	F	F	T	T	T	T

last two columns are the same so we have
$(p_1 \vee p_2) \to q \leftrightarrow (p_1 \to q) \wedge (p_2 \to q)$

5. (15 points)

(a) Let p be the statement "There is a spotty dog that barks at squirrels." Write down the negation of p (i.e $\neg p$) as an English sentence.

p is $\exists d\,(S(d) \wedge B(d))$.

$\neg p$ is $\forall d\,(\neg S(d) \vee \neg B(d))$. ie.

All dogs are either not spotty or don't bark at squirrels.

(b) Find the conjunction of the propositions r and s where r is the proposition "Jim's is taller than 182cm" and s is the proposition "Jim is younger than 23 years old."

$r \wedge s$. is J is taller than 182cm and younger than 23 years old.

(c) Construct a truth table for the proposition $(p \wedge \neg q) \vee (\neg p \wedge q)$.

p	q	$\neg p$	$\neg q$	$p \wedge \neg q$	$\neg p \wedge q$	$(p\wedge\neg q)\vee(\neg p\wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	F	F

6. (20 points)

Prove that $\sqrt{7}$ is not a rational number and explain how your reasoning would break down if you tried to prove that $\sqrt{9}$ is not a rational number.

Assume $\sqrt{7}$ is rational. ie $\sqrt{7} = a/b$ where a & b integers and the fraction a/b cannot be simplified any furthur.

Then $7b^2 = a^2$,

We claim that ~~$7 | a^2$~~

7 dividing a^2 means 7 divides a.

(We prove this by contra positive. ie 7 not dividing a means 7 doesn't divide a^2

7 not dividing $a \Rightarrow a = 7n + i$ where $i = 1, 2, 3, 4, 5, 6$.

$a^2 = (7n+i)^2 = (49n^2 + 14ni + i^2)$

$= 7(7n^2 + 2ni) + i^2$

if $i = 1$, $i^2 = 1$ so 7 doesn't divide a^2 (as of form $7m+1$)

$i = 2$ $i^2 = 4$ so $a^2 = 7m+4$ ie 7 doesn't divide

$i = 3$ $i^2 = 9 = 7 + 2$. so $a^2 = 7(7n^2 + 2ni + 1) + 2 = 7m + 2$

$i = 4$ $i^2 = 14 + 2$, so $a^2 = 7m + 2$.

$i = 5$ $i^2 = 21 + 4$, so $a^2 = 7m + 4$

$i = 6$ $i^2 = 35 + 1$, so $a^2 = 7m + 1$. //

So 7 divides $a \Rightarrow a = 7n$ some n in $\mathbb{Z}$.

ie $a^2 = 49n^2$ so $7b^2 = a^2 = 49n^2$

So $b^2 = 7n^2$ so 7 divides $b^2 \Rightarrow 7$ divides b

So 7 divides a & b ie the fraction a/b can ~~not~~ be simplified. #

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(as we assume a/b simplest form)

This is a contradiction, so we have that $\sqrt{7}$ is irrational.

If we attempt the same thing with $\sqrt{9}$ ~~thee~~ get to

$$9b^2 = a^2.$$

The statement 9 divides $a^2 \Rightarrow$ 9 divides a is false. As 9 divides $3^2 = 9$ but 9 does not divide ~~a~~ 3.