Math 150: Discrete mathematics

Final

Dec 15, 2019

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Instructions:

1. Indicate your instructor with a check in the appropriate box:

Zhang	MW 9:00	
Lorman	MW 10:25	
Mkrtchyan	MW 12:30	
Lubkin	MW 3:25	

2. Read the notes below:

   - The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.
   - Notes of any kind are strictly forbidden.
   - Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
   - You are responsible for checking that this exam has all 19 pages.

3. Read the following Academic Honesty Statement and sign:

   I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature:______________________________

Part I

1. (10 points) Prove that if $x = a \cdot b \cdot c$ where a, b and c are positive real numbers, then either $a \leq x^{1/3}$, $b \leq x^{1/3}$ or $c \leq x^{1/3}$.

Solution Suppose the conclusion is false. Then $a > x^{1/3}$, $b > x^{1/3}$ and $c > x^{1/3}$. Hence $x = a \cdot b \cdot c > x^{1/3} \cdot x^{1/3} \cdot x^{1/3} = x$, $x > x$, a contradiction.

2. (15 points) Recall that if A and B are sets, $A \oplus B$ (also denoted A **XOR** B), the symmetric difference of A and B, is defined to be the set of all x that are in A or B, but not both.

Prove that $A \oplus B = A$ if and only if $B = \emptyset$.

Solution First suppose that $A \oplus B = A$. Suppose for the sake of contradiction that $B \neq \emptyset$. Since $B \neq \emptyset$ we may choose an element x of B. There are two cases: either $x \in A$ or $x \notin A$. If $x \in A$, then $x \in A \cap B$, so x is not in $A \oplus B$ by definition. If $x \notin A$, then since we also have that $x \in B$, we have that $x \in A \oplus B$. But this contradicts the fact that $A \oplus B = A$. Thus, B must be empty.

Now suppose that B is empty. Then $A \cap B = \emptyset$ and $A \cup B = A$. Thus,

$$A \oplus B = (A \cup B) - (A \cap B) = A - \emptyset = A.$$

3. (20 points) Prove each of the following statements from definitions without appealing to any theorems stated in class.

(a) $x^3 + 2x^2 - 6$ is $O(x^3)$.

Solution

For $x > 2$, $x^3 + 2x^2 - 6 < x^3 + x \cdot x^2 + x^3 = 3x^3$. Taking $k = 2$ and $C = 3$, we have that $x^3 + 2x^2 - 6 < Cx^3$ for $x > k$, as required.

(b) x^3 is NOT $O(x^2)$.

Solution

If x^3 is $O(x^2)$, then there are positive integers k and C such that, for all $x > k$ we have that $x^3 < Cx^2$. Dividing by x^2, this gives $x < C$. But if x is any integer greater than both k and C, (e.g., $k+C+1$), it is not true that $x < C$. This is a contradiction. Hence it is false that x^3 is $O(x^2)$.

4. (15 points) Recall the binary search algorithm is given in pseudocode as follows:

procedure *binary search* (x: integer, $a_1, a_2, \ldots, a_n$: increasing integers)
i:=1
j:=n
while $i < j$
  $m := \lfloor (i+j)/2 \rfloor$
  if $x > a_m$ **then** $i := m + 1$
  else j:=m
if $x = a_i$ **then** *location*:=i
else *location:=0*
return *location*

(a) (5pts) What is the exact number of comparisons used by the binary search algorithm for a list with $n = 2^k$ terms which does not contain x (that is, for all i, $a_i \neq x$)? Your answer should be a function of n.

Comparisons:

(b) (10pts) Prove that your answer is correct using mathematical induction on k.

Solution

We solve part (b) first.

Let $f(k)$ equal the number of comparisons needed, for $k \geq 0$.

Base Case: $k = 0$. Then $n = 2^0 = 1$. The algorithm starts by initializing $i = 1$ and $j = n = 1$. The while loop compares i and j, which is 1 comparison. Since it is false the $i < j$, the while loop is exited. The if loop is entered. x and a_1 are compared, a second comparison. The else clause of the if loop is executed, since it is false that $i < j$, location is assigned the value 0, which is then returned.

So when $k = 0$, exactly 2 comparisons are made, so $f(0) = 2$.

Supposed that we have computed $f(k)$ for some integer $k \geq 0$. Then we compute $f(k+1)$ as follows:

We have that $n = 2^{k+1}$. The algorithm starts by initializing $i = 1$ and $j = n = 2^{k+1}$. The while loop compares i and j, which is 1 comparison. Since it is true that $i < j$, the while loop is entered. m is assigned a value, and the if loop is entered. x and a_m are compared, a second comparison. So 2 comparisons are made, and then we are returned to the while loop, but now with a list of 2^k elements. By the inductive hypothesis, it takes $f(k)$ comparisons to complete the algorithm. So $f(k+1) = 2 + f(k)$.

So $f(0) = 2$, $f(k+1) = f(k) + 2$ for $k \geq 1$. Hence $f(k) = 2(k+1)$, for $k \geq 0$.

$n = 2^k$, $k = \log_2(n)$. So the answer to part (a) is: The exact number of comparisons made is $2(\log_2(n) + 1))$.

5. (15 points)

For each of the following congruence equations, either find a solution or show that a solution does not exist:

(a) $36x \equiv 23 \pmod{17}$

Solution

Reducing 36 and 23 modulo 17, the equation simplifies to become

$2x \equiv 6 \pmod{17}$, which has the (in fact, unique) solution $x \equiv 3 \pmod{17}$.

(b) $4x \equiv 2 \pmod{8}$

Solution

If there is a solution, then there is an integer k such that $4x = 2 + 8k$. Dividing by 2, $2x = 1+4k$, $2(x-2k) = 1$, which imples that 1 is an even number, which is false. Hence there is no solution.

(c) $x^{86} \equiv 5 \pmod{87}$

Solution

87 is a multiple of 3. So if x is a solution of the given congruence, then

$x^{86} \equiv 5 \pmod{3}$.

Since $5 \equiv 2 \pmod{3}$, this becomes

$(x^2)^{43} \equiv 2 \pmod{3}$.

The squares of 0, 1 and 2 modulo 3 are congruent to 0, 1 and 1 mod 3. Therefore the last equation implies that

either 0^{43} or $1^{43} \equiv 2 \pmod{3}$; and both of these assertions are false. Hence this congruence has no solutions.

6. (10 points)

Find all odd integers which have the property that they both leave a remainder of 4 when divided by 7 and leave a remainder of 5 when divided by 11.

Solution We seek all integers x such that

$x \equiv 1 \pmod 2$, $x \equiv 4 \pmod 7$, $x \equiv 5 \pmod{11}$.

Let $m_1 = 2$, $m_2 = 7$ and $m_3 = 11$, $M_1 = m_2 \cdot m_3 = 7 \cdot 11 = 77$, $M_2 = m_1 \cdot m_3 = 2 \cdot 11 = 22$, $M_3 = m_1 \cdot m_2 = 2 \cdot 7 = 14$. Let y_i be an inverse of M_i mod m_i for $i = 1, 2, 3$. So $y_1 =$ inverse of 77 mod 2 = inverse of 1 mod 2; so we take $y_1 = 1$. $y_2 =$ an inverse of 22 mod 7 = an inverse of 1 mod 7; so we take $y_2 = 1$. y_3 is an inverse of 14 mod 11 = an inverse of 3 mod 11 = 4; so take $y_3 = 4$.

Then by the CRT, the unique solution mod $2 \cdot 7 \cdot 11$ is $x = 1 \cdot M_1 \cdot y_1 + 4 \cdot M_2 \cdot y_2 + 5 \cdot M_3 \cdot y_3 = 1 \cdot 77 \cdot 1 + 4 \cdot 22 \cdot 1 + 5 \cdot 14 \cdot 4 = 77 + 88 + 280$. This is the solution mod $2 \cdot 7 \cdot 11$, i.e., the unique solution mod 154. $77 + 88 = 165 \equiv 11$ mod 154. $280 \equiv 126$ mod 154. So $x \equiv 11 + 126 = 137$ mod 154.

So the set of all integers x obeying the conditions are all integers that are congruent to 137 mod 154.

7. (5 points)

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

(a) Encrypt the message $TRUE$ using the shift cypher $x \to x + 5 \bmod 26$.

The encrypted message is:
YWZJ

(b) Decrypt the message $ZSVH$ if it was encoded using the shift cypher $x \to x - 5 \bmod 26$.

The decrypted message is:
EXIT

I

Part II

8. (10 points) For this problem do not expand any factorials or binomial coefficients.

(a) Find the coefficient of $x^{20}y^{80}$ in $(3x - 5y)^{100}$.

The coefficient is:
$\binom{100}{80}\ 3^{20}(-5)^{80}$

(b) Find the coefficient of x^{600} in $(2x + 3x^5)^{200}$.

The general term is $\binom{200}{k}\ (2x)^{200-k}(3x^5)^k$. Let $200 - k + 5k = 600$. We have $k = 100$. Thus the coefficient is $\binom{200}{100}\ 2^{100}3^{100}$

The coefficient is:
$\binom{200}{100}\ 2^{100}3^{100}$

9. (15 points) For this problem do not expand any factorials or binomial coefficients. Incorrect answers without justification will not get credit.

How many ways are there for 4 men and 3 women to stand in a line so that

(a) all men stand together?

Number of ways:
$4! \cdot 4!$

(b) all women stand together?

Number of ways:
$5! \cdot 3!$

(c) all men stand together or all women stand together? (inclusive or)
Use inclusion-exclusion

Number of ways:
$4! \cdot 4! + 5! \cdot 3! - 2! \cdot 3! \cdot 4!$

10. (10 points) There are 10 red balls and 10 blue balls. You select balls at random without looking at them.

(a) What is the smallest number of balls you must select to ensure having at least 4 balls of the same color? Justify your answer.

Solve $\lceil \frac{N}{2} \rceil \geq 4$, we have $N \geq 7$.

Number of balls:
7

(b) What is the smallest number of balls you must select to ensure having at least 4 blue balls? Justify your answer.

In the worst case, you may have 10 red balls before you get the first blue ball. So you must select $10 + 4 = 14$ balls to ensure having at least 4 blue balls.

Number of balls:
14

11. (10 points)

(a) (8pts) A strange species has the following life cycle. Each animal lives exactly two years. At the end of two years, it lays one egg, and dies. The egg hatches exactly one year later. You start a farm with one newly laid egg and one newly hatched animal. Let a_n be the number of live animals you have exactly n years after this.

Find a recurrence relation for the sequence a_n. Explain your reasoning. You do not need to solve the recurrence relation.

a_n=the number of 1-year-old animals in the n'th year + the number of 2-year-old animals in the n'th year

Fact 1: The number of 1-year-old animals in the n'th year = the number of eggs in the (n-1)'th year = the number of 2-year-old animals in the (n-2)'th year = the number of 1-year-old animals in the (n-3)'th year

Fact 2: The number of 2-year-old animals in the n'th year = the number of 1-year-old animals in the (n-1)'th year = the number of 2-year-old animals in the (n-3)'th year (use Fact 1)

Thus $a_n = a_{n-3}$

The recurrence relation is:

$a_n = a_{n-3}$

(b) (2pts) How many initial terms do we need so that the recurrence relation $a_n = a_{n-1} - a_{n-3}$ has a unique solution? The degree is $n - (n-3) = 3$. So we need 3 initial terms.

Need:

3

12. (20 points) Solve the recurrence relation

(a) $a_n + 2a_{n-1} + a_{n-2} = 0, \forall n \geq 2$ and $a_0 = 1, a_1 = 7$.

Solve $r^2 + 2r + 1 = 0$ we get $r_1 = r_2 = -1$. Let $a_n = (c_1 + c_2 n)(-1)^n$. Then $a_0 = c_1 = 1, a_1 = -(c_1 + c_2) = 7$. We get $c_1 = 1, c_2 = -8$. Thus $a_n = (1 - 8n)(-1)^n$.

a_n is :
$a_n = (1 - 8n)(-1)^n$

(b) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, \forall n \geq 2$ and $a_0 = 5, a_1 = 2, a_2 = 8$. (Hint: 2 is a root of the characteristic polynomial).

We need to solve $r^3 = 2r^2 + r - 2$. Since 2 is a root, we compute $(r^3 - 2r^2 - r + 2)/(r-2) = r^2 - 1 = (r-1)(r+1)$. Hence $r_1 = 1, r_2 = -1, r_3 = 2$. Let $a_n = c_1 + c_2(-1)^n + c_3 2^n$. Then $a_0 = c_1 + c_2 + c_3 = 5, a_1 = c_1 - c_2 + 2c_3 = 2, a_2 = c_1 + c_2 + 4c_3 = 8$. Then $c_1 = 2, c_2 = 2, c_3 = 1$. So $a_n = 2 + 2(-1)^n + 2^n$

a_n is :

$a_n = 2 + 2(-1)^n + 2^n$

13. (13 points)

(a) (10pts) Ara, Ben, David, Elen and Narek are friends. Someone conducted a survey and asked each one of them with how many of the others the person had a phone conversation in the past week. The answers were as follows: Ara talked with 3 of the friends, Ben with 2, David with 3, Elen with 2 and Narek with 3.

Is this possible? If yes, show how, if not, explain why not.

No. We may use a graph to represent the survey results. There are 5 vertices: Ara, Ben, David, Elen and Narek. The degree of each vertex is the number of friends he/she talked with. Then their degrees are 3,2,3,2,3. The sum of the degrees is $3+2+3+2+3=13$ is odd, which contradicts to the handshaking theorem for the graph.

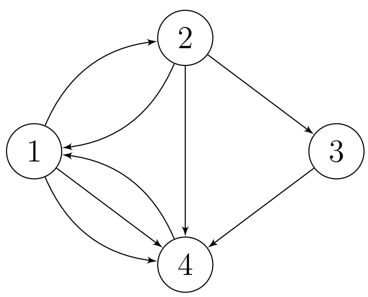
(b) (1pts) Define the complete graph K_n on n vertices.
A simple graph which has n vertices and contains exactly one edge between any two distinct vertices.

(c) (2pts) How many edges does the complete graph K_{100} have?

The number of edges is :
$100 \cdot 99/2 = 4950$

14. (12 points)

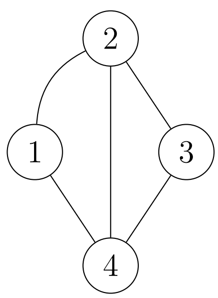
(a) What is the adjacency matrix of the following directed graph, if the vertex labeled i corresponds to row i for $i = 1, 2, 3, 4$? The row positions correspond to the start of each directed edge, and columns to the end of each directed edge.

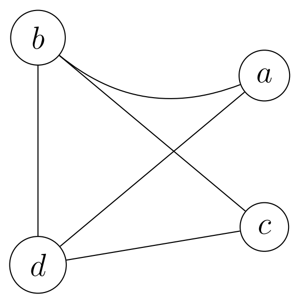


The adjacency matrix is :

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

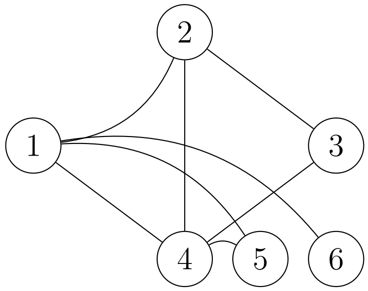
(b) If the following graphs are isomorphic, give an isomorphism. If not, explain why not.

(i) 

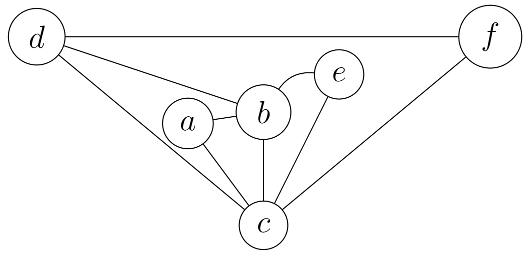
and 

Yes. $1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d$ is an isomorphism.

(ii)

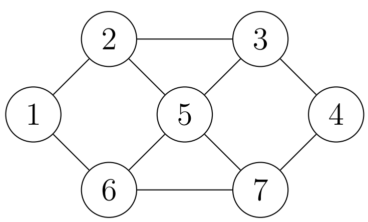


and

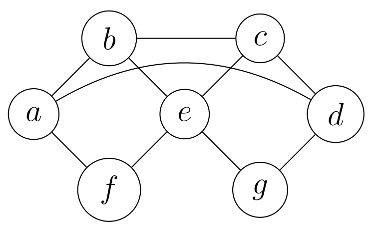


No. Vertex c on the right has degree 5, but no vertex on the left has degree 5.

(iii)



and



No. There are 2 simple circuits of length 3 on the left, but there is exactly 1 simple circuit of length 3 on the right.

You may use this page if you run out of space. Be sure to label your problems on this page and also include a note on the original page telling the graders to look for your work here.