Math 150: Discrete mathematics

Final

Dec 15, 2019

NAME (please print legibly): ____________________

Your University ID Number: ____________________

Instructions:

1. Indicate your instructor with a check in the appropriate box:

Zhang	MW 9:00	
Lorman	MW 10:25	
Mkrtchyan	MW 12:30	
Lubkin	MW 3:25	

2. Read the notes below:

   - The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.
   - Notes of any kind are strictly forbidden.
   - Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
   - You are responsible for checking that this exam has all 18 pages.

3. Read the following Academic Honesty Statement and sign:

   I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature:____________________

Part I

1. (10 points) Prove that if $x = a \cdot b \cdot c$ where a, b and c are positive real numbers, then either $a \leq x^{1/3}$, $b \leq x^{1/3}$ or $c \leq x^{1/3}$.

2. (15 points) Recall that if A and B are sets, $A \oplus B$ (also denoted A **XOR** B), the symmetric difference of A and B, is defined to be the set of all x that are in A or B, but not both.

Prove that $A \oplus B = A$ if and only if $B = \emptyset$.

3. (20 points) Prove each of the following statements from definitions without appealing to any theorems stated in class.

(a) $x^3 + 2x^2 - 6$ is $O(x^3)$.

(b) x^3 is NOT $O(x^2)$.

4. (15 points) Recall the binary search algorithm is given in pseudocode as follows:

procedure *binary search* (x: integer, $a_1, a_2, \ldots, a_n$: increasing integers)
i:=1
j:=n
while $i < j$
  $m := \lfloor (i+j)/2 \rfloor$
  if $x > a_m$ **then** $i := m + 1$
  else j:=m
if $x = a_i$ **then** *location*:=i
else *location:=0*
return *location*

(a) (5pts) What is the exact number of comparisons used by the binary search algorithm for a list with $n = 2^k$ terms which does not contain x (that is, for all i, $a_i \neq x$)? Your answer should be a function of n.

Comparisons:

(b) (10pts) Prove that your answer is correct using mathematical induction on k.

5. (15 points)

For each of the following congruence equations, either find a solution or show that a solution does not exist:

(a) $36x \equiv 23 \pmod{17}$

(b) $4x \equiv 2 \pmod{8}$

(c) $x^{86} \equiv 5 \pmod{87}$

6. (10 points)

Find all odd integers which have the property that they both leave a remainder of 4 when divided by 7 and leave a remainder of 5 when divided by 11.

7. (5 points)

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

(a) Encrypt the message $TRUE$ using the shift cypher $x \to x + 5 \bmod 26$.

The encrypted message is:

(b) Decrypt the message $ZSVH$ if it was encoded using the shift cypher $x \to x - 5 \bmod 26$.

The decrypted message is:

Part II

8. (10 points) For this problem do not expand any factorials or binomial coefficients.

(a) Find the coefficient of $x^{20}y^{80}$ in $(3x - 5y)^{100}$.

The coefficient is:

(b) Find the coefficient of x^{600} in $(2x + 3x^5)^{200}$.

The coefficient is:

9. (15 points) For this problem do not expand any factorials or binomial coefficients. Incorrect answers without justification will not get credit.

How many ways are there for 4 men and 3 women to stand in a line so that

(a) all men stand together?

Number of ways:

(b) all women stand together?

Number of ways:

(c) all men stand together or all women stand together? (inclusive or)

Number of ways:

10. (10 points) There are 10 red balls and 10 blue balls. You select balls at random without looking at them.

(a) What is the smallest number of balls you must select to ensure having at least 4 balls of the same color? Justify your answer.

Number of balls:

(b) What is the smallest number of balls you must select to ensure having at least 4 blue balls? Justify your answer.

Number of balls:

11. (10 points)

(a) (8pts) A strange species has the following life cycle. Each animal lives exactly two years. At the end of two years, it lays one egg, and dies. The egg hatches exactly one year later. You start a farm with one newly laid egg and one newly hatched animal. Let a_n be the number of live animals you have exactly n years after this.

Find a recurrence relation for the sequence a_n. Explain your reasoning. You do not need to solve the recurrence relation.

The recurrence relation is:

(b) (2pts) How many initial terms do we need so that the recurrence relation $a_n = a_{n-1} - a_{n-3}$ has a unique solution?

Need:

12. (20 points) Solve the recurrence relation

(a) $a_n + 2a_{n-1} + a_{n-2} = 0, \forall n \geq 2$ and $a_0 = 1, a_1 = 7$.

a_n is :

(b) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} = 0, \forall n \geq 2$ and $a_0 = 5, a_1 = 2, a_2 = 8$. (Hint: 2 is a root of the characteristic polynomial).

a_n is :

13. (13 points)

(a) (10pts) Ara, Ben, David, Elen and Narek are friends. Someone conducted a survey and asked each one of them with how many of the others the person had a phone conversation in the past week. The answers were as follows: Ara talked with 3 of the friends, Ben with 2, David with 3, Elen with 2 and Narek with 3.

Is this possible? If yes, show how, if not, explain why not.

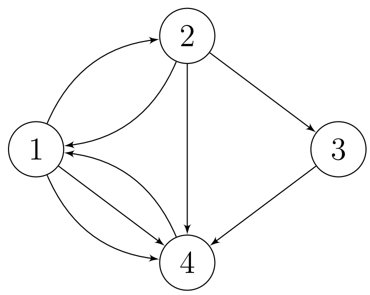
(b) (1pts) Define the complete graph K_n on n vertices.

(c) (2pts) How many edges does the complete graph K_{100} have?

The number of edges is :

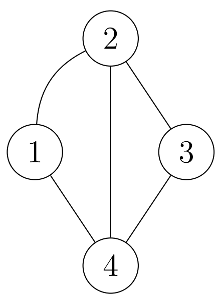
14. (12 points)

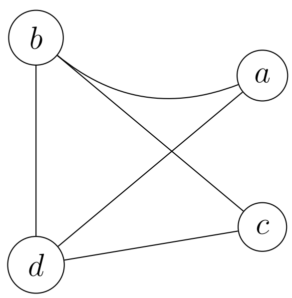
(a) What is the adjacency matrix of the following directed graph, if the vertex labeled i corresponds to row i for $i = 1, 2, 3, 4$? The row positions correspond to the start of each directed edge, and columns to the end of each directed edge.

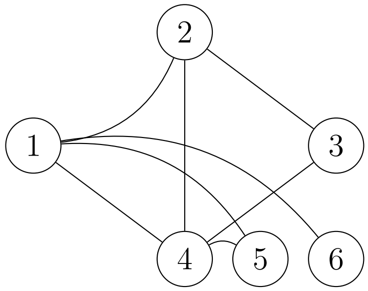


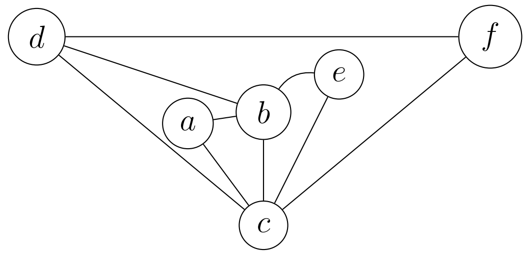
The adjacency matrix is :

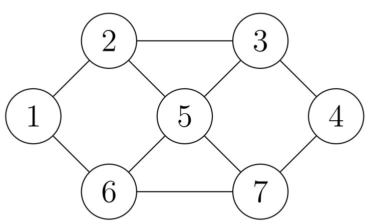
(b) If the following graphs are isomorphic, give an isomorphism. If not, explain why not.

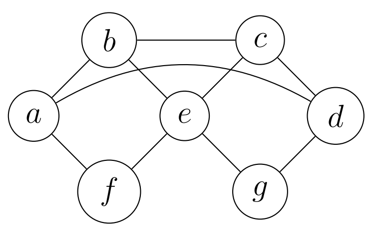
(i) 

and 

(ii) 

and 

(iii) 

and 

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