Math 150: Discrete Mathematics

Midterm 2

November 20, 2018

NAME (please print legibly): Solutions.

Your University ID Number: ______________

Indicate the lecture time you attend with a check in the appropriate box:

S. Amelotte	MW 3:25–4:40pm	
A. Iosevich	MW 10:25–11:40am	
J. Passant	MW 9:00–10:15am	
V. Petkov	MW 12:30–1:45pm	
MTH150A		

- MTH150A students, if you wish the exam returned in a class, please mark that instructor in addition to the MTH150A box.
- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 14 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers.You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Please sign the pledge below.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________

QUESTION	VALUE	SCORE
1	15	
2	20	
3	15	
4	15	
5	15	
6	20	
TOTAL	100	

1. (15 points)

(a) Consider the number 407, written in base 10.

(i) Write 99 in base 2.

$$99 = 2\cdot 49 + 1 = 2\cdot(2\cdot 24+1)+1 = 4\cdot 24 + 2 + 1$$
$$= 4\cdot(2\cdot 12) + 2 + 1$$
$$= 8\cdot 12 + 2 + 1$$
$$= 8\cdot(2\cdot 6) + 2 + 1$$
$$= 16\cdot 6 + 2 + 1$$
$$= 2^4(2\cdot 3) + 2 + 1$$
$$= 2^5(2+1) + 2 + 1$$
$$= 2^6 + 2^5 + 2 + 1.$$

$49 = 2\cdot 24 + 1$

$24 = 2\cdot 12$

$12 = 2\cdot 6$

$6 = 2\cdot 3$

$3 = 2 + 1$

So $99 = (1100011)_2$.

(ii) Write 7798 in hexadecimal.

0 1 2 3 4 5 6 7 8 9 A B C D E F
10 11 12 13 14 15

$7798 = 487\cdot 16 + 6$

$487 = 30\cdot 16 + 7$

$30 = 16 + 14$

$$7798 = (30\cdot 16 + 7)\cdot 16 + 6$$
$$= 30\cdot 16^2 + 7\cdot 16 + 6$$
$$= (16+14)\cdot 16^2 + 7\cdot 16 + 6$$
$$= 16^3 + 14\cdot 16^2 + 7\cdot 16 + 6$$

So $7798 = (1E76)_{16}$.

(iii) Write 3 in base 7.

$3 = 0\cdot 7 + 3$ so $3 = (3)_7$.

(d) Find $(2AE01)_{16} + (AA1)_{16}$, giving your answer in base 16.

$$\begin{array}{r} 2A\,E01 \\ AA1 \\ \hline 25A2 \quad \text{base } 10 \\ (18)_{16}A2 \quad \text{base } 16 \\ \hline 2B8A2 \end{array}$$

$E = 14$
$A = 10$.

$24 = 16 + 8$

$A + 1 = 11 = B$.

$$\boxed{(2B8A2)_{16}.}$$

(e) Find $(222)_3 \times (27)_9$, giving your answer in base 3 or base 9.

$(222)_3 = 2 \cdot 3^2 + 2 \cdot 3 + 2 = 18 + 6 + 2 = 26$

$(28)_9 = 2 \cdot 9 + 8 = 18 + 8 = 26$

$101 = 99 + 2$
$= 9 \cdot 11 + 2.$

$26^2 = 400 + 480 + 36 = 916 = 100 \cdot 9 + 16$
$= 101 \cdot 9 + 7$
$= (9 \cdot 11 + 2) \cdot 9 + 7$
$= 9^2 \cdot 11 + 2 \cdot 9 + 7$
$= 9^2(9+2) + 2 \cdot 9 + 7$

So $916 = (1227)_9$. //

2. (20 points)

Find x such that $0 \leq x < 450$ and

$$x \equiv 0 \mod 2,$$
$$x \equiv 24 \mod 25,$$
$$x \equiv 6 \mod 9.$$

We now $x = a_1 M_1 S_1 + a_2 M_2 S_2 + a_3 M_3 S_3$

Where $a_1 = 0$, $a_2 = 24 \equiv -1 \bmod 25$, $a_3 = 6$

$M_1 = 25 \cdot 9 = 225$, $M_2 = 18$, $M_3 = 50$.

S_1 the inverse of 225 mod 2.

$225 \equiv 1 \bmod 2$ so has inverse 1

ie $S_1 = 1$.

S_2 inverse of 18 mod 25 using Euclid:

$25 = 18 + 7$
$18 = 2 \cdot 7 + 4$
$7 = 4 + 3$.
$4 = 3 + 1$

$1 = 2 \cdot 18 - 5 \cdot (25 - 18)$
$1 = 2 \cdot (18 - 2 \cdot 7) - 7 = 2 \cdot 18 - 5 \cdot 7$
$1 = 4 - (7 - 4) = 2 \cdot 4 - 7$
$1 = 4 - 3$

So $1 = \cancel{7} \cdot 18 - 5 \cdot 25$ So $S_2 = 7$.

S_3 the inverse of 50 mod 9 $50 \equiv 5 \bmod 9$

$5 \cdot 1 \equiv 5$, $5 \cdot 2 \equiv 10 \equiv 1 \bmod 9$ so $S_3 = 2$.

18
36
180
90 = 5·18
36
126

$x = 6 \cdot 225 \cdot 1 + (-1) \cdot 18 \cdot 7 + 6 \cdot 50 \cdot 2$

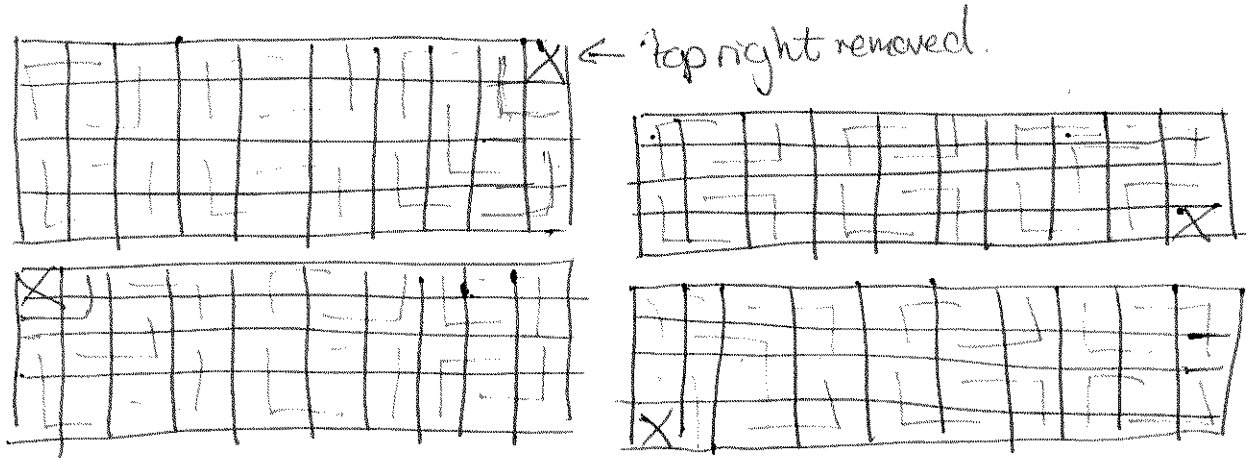
$= -126 + 600 = 474 \equiv 24 \bmod 450$.

Note $24 \equiv 0 \bmod 2$, $24 \equiv 24 \bmod 25$, $24 \equiv 6 \bmod 9$, so it works.

3. (15 points)

(a) Let n be any positive integer. Prove that one can tile a $10^n \times 4^n$ chess board with exactly one of the four corner tiles removed (could be any of the four) only using the right triominos. [*Hint: There should be four base cases.*]

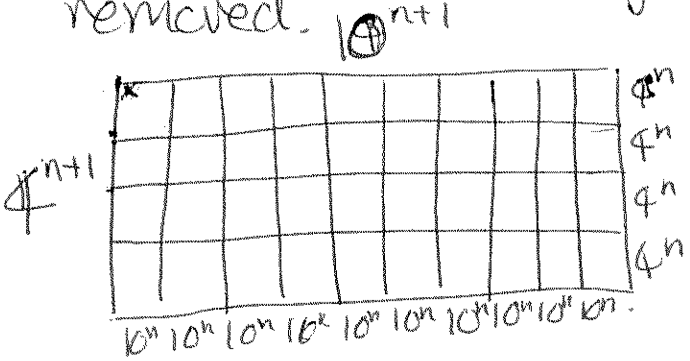
Base cases $n=1$ so a 4×10 board.



So works in all four base cases.

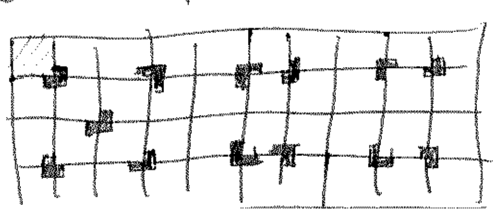
Induction step

We take a $4^{n+1}\times 10^{n+1}$ grid and assume that we can tile an $4^n \times 10^n$ grid with any of the four corners removed.



we divide our $4^{n+1}\times 10^{n+1}$ grid up into 4×10 grids of size $4^n \times 10^n$. we assume the top left tile is removed (other cases are the same).

We can now tile the upper left corner $4^n \times 10^n$ grid using the induction hypothesis (as this has a tile removed. We then place trionimos on the grid to form this pattern.



These only cover the corner peices and leave $4^n \times 10^n$ grids with one corner tile removed. Thus we can tile these grids by the I.H.

(b) Prove that you can make any stamp amount of 30 cents or above using only 6 cents and 7 cent stamps.

Base cases

We can make 30, 31, 32, 33, 34 and 35¢ stamps using

$30 = 6+6+6+6+6.$
$31 = 6+6+6+6+7$
$32 = 6+6+6+7+7.$
$33 = 6+6+7+7+7.$
$34 = 6+7+7+7+7$
$35 = 7+7+7+7+7.$

Induction step If we can make a k¢ stamp then by adding a 6¢ stamp we can make a ~~56~~ k+6¢ stamp. Let P(k) be "we can make a k¢ stamp out of 6¢ & 7¢ stamps".

The above statement says $P(k) \Rightarrow P(k+6)$

So we have (by base cases)

$P(30) \Rightarrow P(36) \Rightarrow P(42) \Rightarrow \cdots$
$P(31) \Rightarrow P(37) \Rightarrow P(43) \Rightarrow \cdots$
$P(32) \Rightarrow P(38) \Rightarrow P(44) \Rightarrow \cdots$
$P(33) \Rightarrow P(39) \Rightarrow P(45) \Rightarrow \cdots$
$P(34) \Rightarrow P(40) \Rightarrow P(46) \Rightarrow \cdots$
$P(35) \Rightarrow P(41) \Rightarrow P(47) \Rightarrow \cdots$

Using this we can see we get $P(n)$ $\forall n \geq 30$. //

4. (15 points)

(a) What is 43 mod 21?

$\cancel{43 = 1 \bmod 21}$ $43 \equiv 1 \bmod 43$.

(as $43 = 2 \cdot 21 + 1$)

(b) Find 43^{230} mod 21.

$43^{230} \bmod 21 \equiv 1^{230} \bmod 21 \equiv 1 \bmod 21$.

(c) Find x such that $0 \leq x < 13$ such that $x^2 \equiv -1$ mod 13.

let $x = 5$ $5^2 = 25 \equiv -1 \bmod 13$.

(d) Show that if $\gcd(x, p) = 1$ then there is an integer y such that $xy \equiv 1 \mod p$.

Using Bézout's Theorem as $\gcd(x,p)=1$

$\exists s,t$ integers s.t.

$xs + pt = 1$.

ie $pt = 1 - xs$

so $p \mid 1 - xs \iff 1 - xs \equiv 0 \mod p$

$\iff 1 \equiv xs \mod p$.

so let $s = y$ and we have

$xy \equiv 1 \mod p$. //

5. (15 points)

(a) Define what it means for $f(x)$ to be $\Theta(g(x))$. Your answers should feature four constants C_1, C_2 and k_1, k_2.

(1) We have that $\exists C_1, k_1$ s.t.
$|f(x)| \leq C_1|g(x)|$ for $x \geq k_1$

and

(2) $\exists C_2, k_2$ s.t.
$|f(x)| \geq C_2|g(x)|$ for $x \geq k_2$.

If both (1) & (2) hold $f(x)$ is $\Theta(g(x))$.

(b) Show that $x^2 - 6x + 7$ is $\Omega(x)$, please note the values you use for C and k.

If $x \geq 7$ then $x^2 \geq 7x$ so

$x^2 - 6x + 7 \geq 7x - 6x + 7 = x + 7 \geq x$.

So $C = 1$, $k = 7$.

(c) Show that $x^3 - 6x^2 + 12x - 8$ is $O(x^3)$.

$x^3 - 6x^2 + 12x - 8 \leq x^3 + 6x^2 + 12x + 8$
$\leq x^3 + 6x^3 + 12x^3 + 8x^3$

As $1 \leq x \leq x^2 \leq x^3$ when $x \geq 1$. So,

$x^3 - 6x^2 + 12x - 8 \leq 27x^3$

$k = 1$, $C = 27$.

(d) Show $\log_2(x)$ is $\Theta(\log_{10}(x))$.

$$\log_2(x) = \frac{\log_{10}(x)}{\log_2(10)} \quad \text{let } c = \frac{1}{\log_2(10)}$$

$$\log_2(x) = c \cdot \log_{10}(x), \text{ so } \log_2(x) \le c \cdot \log_{10}(x).$$

$$\log_2(x) \geq c \log_{10}(x).$$

So we have $\log_2(x) = \Theta(\log_{10}(x))$.

(e) Show that x is not $\Omega(x\log(x))$.

If x is $\Omega(x\log(x))$ $\exists$ c, k s.t.

$$x \geq c\,x\log(x) \quad \forall x \geq k.$$

$$\Leftrightarrow \frac{1}{c} \geq \log(x) \quad \forall x \geq k$$

$$\Leftrightarrow e^{1/c} \geq x \quad \forall x \geq k.$$

But c is constant so let $x = \max\{k, e^{1/c}+1\}$

then $x > e^{1/c}$ which contradicts this $\times$

So x is not $\Omega(x\log(x))$.

6. (20 points)

For this question you may assume the fundamental theorem of arithmetic:

> **Theorem** (Fundamental Theorem of Arithmetic, FTA)
> Every integer greater than 1 can be written uniquely as the product of primes (up to the reordering of the primes in the product).

(a) Let a and b be two integers. Define both the greatest common divisor (gcd) of a and b.

gcd(a, b) is the largest ~~number~~ integer which divides both a and b.

(b) Write the $\gcd(a, b)$ in terms of the primes which divide a or b. [*Hint: First write out that $a = p_1 \cdots p_n$, $b = q_1 \cdots q_m$ where p_i, q_j are primes.*]

$a = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$ $\quad b = p_1^{f_1} p_2^{f_2} \cdots p_n^{f_n}$

(it is easier to use the same notation for a fixed prime)

$e_i ; f_i$ repenesents the power of p_i in a & b resp.

$f_i, e_i \geq 0$.

e.g. $98 = 2^2 \cdot 23$ $\quad 207 = 3^2 \cdot 23$

so if $p_1 = 2$, $p_2 = 3$ $p_3 = 23$
then $e_1 = 2, f_1 = 0$
$e_2 = 0, f_2 = 2$
$e_3 = 1 \quad f_3 = 1$.

$\gcd(a, b) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$

where $\alpha_i = \min\{e_i, f_i\}$.

So $\gcd(98, 207) = 23$.

(c) Show that if p is a number such that

$$\text{If } p \mid ab \quad \text{then} \quad p \mid a \text{ or } p \mid b,$$

then p must be prime. [*Hint: Prove the contrapositive i.e. start with p being not prime and show that it cannot satisfy the condition. It will be useful to remember how to negate a logical or.*]

The contrapositive of this is that

p not prime $\Rightarrow \neg(p|ab \Rightarrow (p|a \vee p|b))$

$\equiv$ p not prime $\Rightarrow (p|ab \wedge p\nmid a \wedge p\nmid b)$.

So if we assume p is prime, we need to find an a & b for which p divides the product ab, but p doesn't divide either a or b.

If p is not prime, then $p = xy$ for some integers x, y where $1 < x < p$ and $1 < y < p$ (as p is a composite number).

Now $p \mid xy$ (as p divides itself) but $p \nmid x$ and $p \nmid y$, as $p|x \Rightarrow p \leq x$ which contradicts that $1 < x < p$. Similarly $p \nmid y$. //

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