[10 points](a) Make a truth table to display the truth values of the following compound proposition:

$$[(q \wedge r) \vee \neg p] \rightarrow [(p \vee q) \wedge \neg r]$$

Is it a tautology? **Show all your work.**

p	q	r	$q \wedge r$	$(q\wedge r)\vee(\neg p)$ "L"	$p \vee q$	$(p\vee q)\wedge\neg r$ "R"	$L \rightarrow R$
T	T	T	T	T	T	F	F
T	T	F	F	F	T	T	T
T	F	T	F	F	T	F	T
T	F	F	F	F	T	T	T
F	T	T	T	T	T	F	F
F	T	F	F	T	T	T	T
F	F	T	F	T	F	F	F
F	F	F	F	T	F	F	F

↑ NOT a tautology

[10 points](b) Recall the following rules of inference:
(Modus Ponens:) $[p \wedge (p \rightarrow q)] \rightarrow q$.
(Modus Tollens:) $[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$.
(Hypothetical Syllogism:) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$.
(Disjunctive Syllogism:) $[(p \vee q) \wedge (\neg q)] \rightarrow p$.

$H(x)$: "x is healthy"
$T(x)$: "x tastes good"
$J(x)$: "Jim eats x"
$x \in \{\text{Foods}\}$

Suppose you have the following hypothesis:

$\forall x\, H(x) \rightarrow \neg T(x)$ (H1:) "All foods that are healthy to eat do not taste good."
$H(Tofu)$ (H2:) "Tofu is healthy to eat."
$\forall x\, J(x) \rightarrow T(x)$ (H3:) "Jim only eats what tastes good."
$J(cheeseB)$ (H4:) "Jim eats cheeseburgers."

Explain how you can conclude that "Cheeseburgers are not healthy" and "Jim does not eat Tofu" from only the hypothesis and the basic rules of inference.

H4: $J(cheeseBurger)$
instance H3: $J(cheeseBurger) \rightarrow T(cheeseBurger)$
$\therefore T(cheeseburger)$ (Modus Ponens)
instance H1: $H(cheeseBurger) \rightarrow \neg T(cheeseburger)$
$\therefore \neg H(cheeseBurger)$ (Modus Tollens?)

H2: $H(Tofu)$
H1 instance: $H(Tofu) \rightarrow \neg T(Tofu)$
$\therefore \neg T(Tofu)$ Modus Ponens
H3 instance: $J(Tofu) \rightarrow T(Tofu)$
$\therefore \neg J(Tofu)$ Modus Tollens

2. (20 points) Determine the truth value of each of the following propositions if the universe of discourse for all variables is the set of integers, $\mathbb{Z}$.

[4 points](a) $\exists n(n^2 < 0)$.

F

$n^2 \geq 0$ always when $n \in \mathbb{Z}$

[4 points](b) $\exists m \forall n(mn = n)$.

T

$m = 1$

[4 points](c) $\forall m \exists n(n^2 < m)$.

F

$m = 0$, $\nexists n$, $n^2 < 0$

[4 points](d) $\exists n \forall m(n^2 < m)$.

F

There is no integer s.t all integers m are greater than it.

[4 points](e) $\exists m \exists n((n + m = 4) \wedge (n - m = 1))$.

F

$$\begin{array}{r} n+m=4 \\ n-m=1 \\ \hline 2n=5 \end{array}$$

$\rightarrow n = 5/2 \notin \mathbb{Z}$

$\rightarrow m = \frac{3}{2} \notin \mathbb{Z}$

3. (20 points) (a) The universal set for this problem will be $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $A = \{1, 3, 5\}$, $B = \{3, 7, 8\}$, $C = \{5, 6\}$.

List the elements in each of the following sets:

[3 points](i) $A \cap \bar{B}$

$\{1,5\}$

[3 points](ii) $(A - C) \cup B$

$\{1,3,7,8\}$

[3 points](iii) $B \times C$

$\{(3,5),(3,6),(7,5),(7,6),(8,5),(8,6)\}$

[8 points](b) Write pseudocode for an algorithm which takes as input a list of distinct integers $a_1, \ldots, a_n$ and outputs the **largest positive** integer on this list.

procedure FINDLARGESTPOS ($a_1, \ldots, a_n$; distinct int)
location := 0
M := 0
For $i = 1$ to n
If $a_i > M$ then ($M := a_i$ and location := i)
If location := 0 then output ("no positive integers in list")
otherwise output ($a_{location}$)

[3 points](c) Give a brief explanation in words of how the binary search algorithm searches for an input x in an ordered list of distinct integers. Be complete and precise enough so that an intelligent reader can carry out the algorithm in full with your explanation.

$\{a_1, \ldots, a_n\}$

at each step list is halved $m = \lfloor \frac{n}{2} \rfloor$

$\{a_1, \ldots, a_m\}$ $\{a_{m+1}, \ldots, a_n\}$

x is compared to largest thing in smaller half, if bigger, right half is kept, if smaller, left half is kept. Keep halving lists in this way till arrive at list with only one element. Check if x is this element or not!

1. (20 points) Find the smallest integer k such that each of the following functions is $O(n^k)$. **Show enough work so that it can be understood how you arrived at your answer.**

[7 points](a) $(n^3 + nlog(n))(n^5 + 3n + 1) + (12nlog(n) + 10n)(n^4 + 3)$

$$\underbrace{\underbrace{(n^3 + nlog(n))}_{O(n^3)}\underbrace{(n^5 + 3n + 1)}_{O(n^5)}}_{O(n^3 \cdot n^5) = O(n^8)} + \underbrace{\underbrace{(12nlog(n) + 10n)}_{O(n\log n)}\underbrace{(n^4 + 3)}_{O(n^4)}}_{O(n^5 \log n)}$$

$O(n^8)$ so $k=8$

[7 points](b) $\frac{n^7 log(n)}{n^2+1}$ $= \frac{O(n^8)}{O(n^2)} = O(n^6)$ $k=6$

[6 points](c) $1^{11} + 2^{11} + 3^{11} + 4^{11} + \cdots + n^{11}$ $= O(n^{12})$ $k=12$

as $1^{11} + \cdots + n^{11} \le \underbrace{n^{11} + \cdots + n^{11}}_{n} = n \cdot n^{11} = n^{12}$

5. (20 points)

[7 points](a) Let n be an integer. Prove that $9n + 5$ even implies n odd.

Proof by contraposition. Will prove equivalent contrapositive n even $\longrightarrow$ $9n+5$ odd

Pf: n even $\longrightarrow n = 2k$, k int

$\longrightarrow 9n+5 = 9(2k)+5 = 18k+5 = 2(\underbrace{9k+4}_{s \text{ int}})+1$

$= 2s+1 = \text{odd}$ ✓

[7 points](b) Prove that there is no positive integers m and n that solve the equation $2n^2 + 5m^2 = 14$.

Pf: $2n^2+5m^2=14 \longrightarrow 2n^2 \leq 14$ & $5m^2 \leq 14$

$\longrightarrow n^2 \leq 7$ & $m^2 \leq \frac{14}{5} = \frac{28}{10} = 2.8$

$\longrightarrow n \in \{0, \pm 1, \pm 2\}$, $m \in \{0, \pm 1\}$

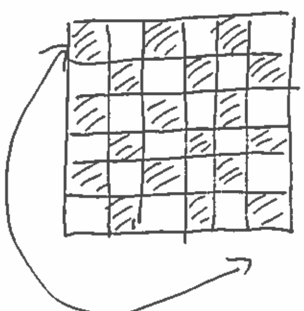
So remains to check these cases:

m \ n	0	±1	±2
0	0	2	8
±1	5	7	13

← value of $2n^2+5m^2$ in boxes

none is 14 so $2n^2+5m^2=14$ has no integer solutions n,m!

[6 points](c) Can a 6×6 chessboard with 2 corners on diagonally opposite sides removed be tiled with standard 2×1 dominoes? If your answer is yes, provide a tiling ; if your answer is no, prove that no tiling exists.



Opposite corners have same color (say black WLOG)

So after removing them have $\frac{1}{2}(6 \times 6) - 2 = 16$ black

$\frac{1}{2}(6 \times 6) = 18$ white

$16 \neq 18$ so impossible to tile by dominoes as each domino must cover one W + one B square and so tilings cover equal # of both.