Math 150: Discrete Mathematics

Practice Final

December 16, 2018

NAME (please print legibly): Solutions

Your University ID Number: ________________

Indicate the lecture time you <u>attend</u> with a check in the appropriate box:

S. Amelotte	MW 3:25–4:40pm	
A. Iosevich	MW 10:25–11:40am	
J. Passant	MW 9:00–10:15am	
V. Petkov	MW 12:30–1:45pm	
MTH150A		

- MTH150A students, if you wish the exam returned in a class, please mark that instructor in addition to the MTH150A box.
- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 18 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers.You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Please sign the pledge below.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ________________

Part A		
QUESTION	VALUE	SCORE
1	10	
2	20	
3	10	
4	15	
5	15	
6	15	
7	15	
TOTAL	100	

Part B		
QUESTION	VALUE	SCORE
8	15	
9	15	
10	15	
11	15	
12	20	
13	20	
TOTAL	100	

Part A

1. (10 points)

(a) Define the sets $A \cup B$ and $\overline{A}$ using the sets A and B.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

$$\overline{A} = \{x \mid x \notin A\}.$$

(b) Let A, B be sets. Prove the identity

$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$

$$\overline{A \cap B} = \{x \mid x \notin A \cap B\}.$$

$$= \{x \mid \neg(x \in (A \cap B))\}$$

$$= \{x \mid \neg((x \in A) \wedge (x \in B))\}$$

DeMorgan

$$= \{x \mid (\neg(x \in A)) \vee (\neg(x \in B))\}$$

$$= \{x \mid x \notin A \vee x \notin B\}$$

$$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$$

$$= \overline{A} \cup \overline{B}.$$

2. (20 points)

(a) Find the number s such that $0 < s \leq 26$ and $13s \equiv 1 \bmod 27$.

$$27 = 2 \cdot 13 + 1$$

So $1 = 27 - 2 \cdot 13$.

So $s = -2$ will work, but it isn't in our range, so $-2 \equiv -2 + 27 \pmod{27}$

$-2 \equiv 25 \pmod{27}$

$$13 \cdot 25 \equiv 1 \pmod{27}$$

[We are using that if $\gcd(a,m) = 1$ then we can find s, t s.t $1 = as + tm$ (Bézout). If this is the case then $as \equiv 1 \bmod m$.]

(b) Find a number t such that $0 < t \leq 26$ and $13t \equiv 15 \bmod 27$.

$13 \cdot 25 \equiv 1 \bmod 27$

So $13 \cdot 25 \cdot 15 \equiv 15 \bmod 27$

$\Rightarrow t \equiv 25 \cdot 15 \bmod 27$

$25 \cdot 15 = 250 + 125 = 175 = 6 \cdot 27 + 13$

So $175 \equiv 13 \bmod 27$ so $t = 13$ works.

270
135 $5 \cdot 27$
27
162

3. (10 points) Consider the following algorithm.

procedure $f(a_1, \ldots, a_n$: non-negative integers with $n \geq 2)$
for $i = 1$ **to** $n - 1$
  for $j = 1$ **to** $n - i$
    if $a_j > a_{j-1}$ **then** interchange a_j and a_{j-1}

(a) What is the specific name of the method the above algorithm is preforming.

Bubble sort.

(b) Would the above algorithm work if $a_1, \ldots, a_n$ were real numbers (as supposed to integers).

Yes. The only restriction is that $a_j > a_{j-i}$ works, which is true for real numbers.

4. (15 points)

(a) State the principle of mathematical induction.

To prove $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps.

(1) We verify $P(1)$ is true.

(2) We show the conditional statement $P(k) \to P(k+1)$ is true for ~~general k~~. all pos. integers k.

(b) Prove by induction that

$$1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{(n+1)} - 1}{2}.$$

Base case $\quad 1+3 = 4 \qquad \frac{3^2-1}{2} = \frac{9-1}{2} = \frac{8}{2} = 4$

So works for $n=1$.

Induction Step We assume $P(k)$ ie "$\underbrace{1+3+\dots+3^k = \frac{3^{k+1}-1}{2}}_{\text{I.H.}}$".

Then

$$1+3+\dots+3^k+3^{k+1} \overset{\text{I.H.}}{=} \frac{3^{k+1}-1}{2} + 3^{k+1}$$

$$= \frac{3^{k+1}-1}{2} + \frac{2\cdot 3^{k+1}}{2}$$

$$= \frac{3^{k+1}+2\cdot 3^{k+1}-1}{2}$$

$$= \frac{3\cdot 3^{k+1}-1}{2} = \frac{3^{k+2}-1}{2}. \quad //$$

5. (15 points)

(a) Show that if $\gcd(a, m) = 1$ and $ab \equiv ac \pmod{m}$, then $b \equiv c \pmod{m}$.

We first show that $(m | ab \wedge \gcd(a,m) = 1) \Rightarrow m | b$)

Proof $\gcd(a,m) = 1$ means by Bézout that $1 = as + bt$.

$m|ab$ means ~~$\exists n$ s.t.~~ int. n s.t. $ab = mn$.

$\therefore$ $b = b(as + mt) = bas + btm = mns + btm$

$b = m\underbrace{(ns + tb)}_{k} \Rightarrow mk = b \Leftrightarrow m | b$. //

$ab \equiv ac \pmod{m} \Leftrightarrow ab - ac \equiv 0 \pmod{m} \Leftrightarrow m | (ab - ac)$
$\Leftrightarrow m | a(b-c)$. As $\gcd(a,m) = 1$, by above we have that $m | (b-c) \Leftrightarrow b - c \equiv 0 \pmod{m} \Leftrightarrow b \equiv c \pmod{m}$.

(b) Does $ab \equiv ac \pmod{m}$ imply that $b \equiv c \pmod{m}$ without the condition $\gcd(a, m) = 1$? *Either prove or give an explicit counter example.*

No. Let $m = 6$.

$2 \cdot 3 \equiv 0 \mod 6$.

$2 \cdot 6 \equiv 0 \mod 6$ so

$2 \cdot 3 \equiv 2 \cdot 6 \mod 6$ but $2 \not\equiv 6 \mod 6$.

So we have a counter example.

6. (15 points)

(a) Suppose that p is a prime how that if n and m are squares mod p (i.e. the is some x such that $x^2 = n \bmod p$), then nm is also a square mod p.

$n \equiv x^2 \bmod p \quad m \equiv y^2 \bmod p$

So $nm \equiv x^2 y^2 \equiv (xy)^2 \bmod p$.

So nm is a square mod p.

(b) Is this true if p is not prime? *If yes prove it, if no give an explicit counter example.*

Yes, the same proof holds as above.

(c) Suppose that p is a prime. Show that is n is a square mod p and k is an integer such that $nk \equiv 1 \bmod p$, then k is a square also.

If $n = 0$ then no such k, so assume $n \neq 0$.

$n \equiv x^2 \bmod p$. ~~Then~~

~~$nk \equiv x^2 k \bmod p$.~~

Then as p prime $\gcd(x, p) = 1$ so $\exists s$ s.t.
$xs \equiv 1 \bmod p$.

Then $1 \equiv nk \equiv x^2 \cdot k \bmod p$.

$s^2 \equiv s^2 x^2 k \bmod p \quad sx \equiv 1$ so $s^2 x^2 \equiv 1 \bmod p$

So $s^2 \equiv k \bmod p$. //

Part B

7. (15 points) *Throughout the question you may leave your answer in terms of factorials.*

(a) If one has 16 basketball teams in a tournament which requires the teams are first put into 4 groups (of 4 teams). How many ways are there to from the first group?

$\binom{16}{4}$ as 16 teams, we are choosing 4.

$$\binom{16}{4} = \frac{16!}{12!\,4!}$$

(b) Given the first group has been picked, how many ways are there to form the second group?

$\binom{12}{4} = \frac{12!}{8!\,4!}$ As only 12 teams remain,

(c) Thus, how many distinct ways are there to set up all four groups?

$$\underbrace{\binom{16}{4}}_{\text{group 1}} \cdot \underbrace{\binom{12}{4}}_{\text{gp. 2}} \cdot \underbrace{\binom{8}{4}}_{\text{gp3}} \cdot \underbrace{1}_{\text{gp. 4.}} = \frac{16!\,\cancel{12!}\,\cancel{8!}}{\cancel{12!}\,\cancel{8!}\,4!\,4!\,4!\,4!} = \frac{16!}{(4!)^4}.$$

8. (15 points) Consider the following linear cypher

$$f : \mathbb{Z}_{26} \to \mathbb{Z}_{26} \quad \text{such that} \quad f(p) = p - 7.$$

Recall that we encode the letters as

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

(a) Encrypt the message "HELLO WORLD".

HELLO WORLD

AXEEH PHKEW.

(b) Decrypt the message "ABBZNXLL".

ABBZNXLL

HIIGUESS.

9. (15 points)

(a) Prove that $P(n,k)$ the number of ways of choosing ordered lists of k indistinguishable objects out of n, $1 \leq k \leq n$, is equal to $\frac{n!}{(n-k)!}$.

We have n ways to choose the first element of our ordered list, n-1 to choose the 2nd element of our ordered list, etc... until we ~~choose~~ have $n-(k-1)$ choices for the k^{th} element. Thus.

$$P(n,k) = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-(k-1)) = \frac{n!}{(n-k)!}$$

(b) How many permutations of the string "ABCDEFGHIJ" contain the string "ACDE". *You may leave your answer in terms of factorials.*

We use the blocks.

ACDE, B, F, G, I, I and J.

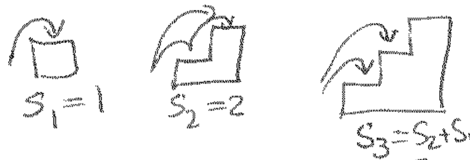
We care about ordered lists of these 7 blocks, which involve all 7 blocks. So there are

$P(7,7) = 7!$ such strings.

10. (15 points) A rabbit can hop up one or two stairs at a time.

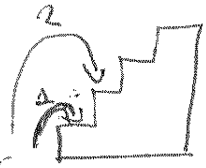
(a) In how many ways can the rabbit climb a staircase with 5 stairs?

Let S_n be the number of ways to hop up n stairs.



$S_1 = 1$ $S_2 = 2$ $S_3 = S_2 + S_1 = 3$.

Two ways to start
(1) we take a 1 hop. Then two steps remaining, so $S_2 = 2$ ways to finish.
(2) we take 2 hop.
Then 1 step remaining, so $S_1 = 1$ way to finish.



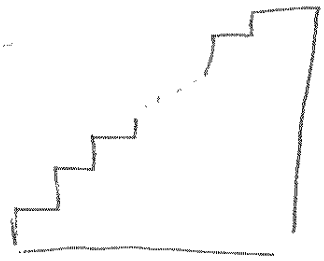
Again two ways to start.
If a 1 hop, then we have 3 steps left so $S_3 = 3$ ways to finish
If a 2 hop, then 2 steps left, so $S_2 = 2$ ways
So $S_4 = S_2 + S_3 = 3 + 2 = 5$.

(b) What about a staircase with n stairs? Here n is any positive integer.

cont.

Finally, we have $S_5 = S_4 + S_3 = 5 + 3 = 8$.

(b)



n stairs

If we start with a 1 hop, then $n-1$ steps left to count. There are S_{n-1} ways to do this.
If start with a 2 hop $n-2$ steps remaining so S_{n-2} ways.

So $S_n = S_{n-1} + S_{n-2}$.

Letting $S_n = r^n$ we get characterist poly $r^2 - r - 1 = 0$ which has roots $r = \frac{1 \pm \sqrt{5}}{2}$. So $S_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$. $S_1 = 1, S_2 = 2$.

So, $1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)$, $2 = \alpha_1\left(1 + \frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(1 + \frac{1-\sqrt{5}}{2}\right)$

So $1 = \alpha_1 + \alpha_2$
$\alpha_1 = 1 - \alpha_2$

$$1 = (1-\alpha_2)\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)$$
$$= \left(\frac{1+\sqrt{5}}{2}\right) - \sqrt{5}\,\alpha_2 \Rightarrow \alpha_2 = \frac{1-\sqrt{5}}{2\sqrt{5}} = \frac{\sqrt{5}-5}{10}$$

So $\alpha_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$

$$S_n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

(see question 12 for a more detailed explanation & solving $S_n = S_{n-1} + S_{n-2}$.)

11. (15 points)

(a) State the pigeon hole principle for n objects and k boxes.

If we have n objects put into k boxes, then there is a box with at least $\lceil n/k \rceil$ objects in it.

(b) What is the minimum number of integers required to ensure that 7 of the chosen integers are odd or 7 of the integers are even. *If no such integer exists then state this.*

There are two boxes $\underset{\text{odd}}{\sqcup}$, $\underset{\text{even}}{\sqcup}$.

If we want 7 objects in one of these two boxes we need smallest n s.t. $\lceil n/2 \rceil = 7$. This ~~is clea~~ n is going to be 13.

So we need 13 integers.

(c) What is the minimum number of integers required to ensure that 7 of the chosen integers are odd. *If no such integer exists then state this.*

No such number exists

(As we could choose infinitely many even integers without getting 7 odd ones.)

12. (20 points)

(a) Solve the recurrence

$$a_n = -a_{n-1} + 6a_{n-2};\ a_0 = 3, a_1 = 1.$$

We use $a_n = r^n$ to get

$$r^n = -r^{n-1} + 6r^{n-2} \Leftrightarrow r^n + r^{n-1} - 6r^{n-2} = 0$$
$$\Leftrightarrow r^{n-2}(r^2 + r - 6) = 0$$
$$\Leftrightarrow r = 0 \text{ (not true)}$$
$$\text{or } r^2 + r - 6 = 0.$$

We solve $r^2 + r - 6 = 0 \Leftrightarrow (r+3)(r-2) = 0$

Let $r_1 = -3$, $r_2 = 2$.

Then $a_n = \alpha_1(-3)^n + \alpha_2(2)^n$. $a_0 = 3$, $a_1 = 1$

We need to find α_1 and α_2. We use a_0 & a_1

$$3 = a_0 = \alpha_1(-3)^0 + \alpha_2(2)^0 = \alpha_1 + \alpha_2.$$
$$1 = a_1 = \alpha_1(-3)^1 + \alpha_2(2)^1 = 2\alpha_2 - 3\alpha_1$$

$$\left.\begin{aligned} 3 &= \alpha_1 + \alpha_2 \\ 1 &= 2\alpha_2 - 3\alpha_1 \end{aligned}\right\}$$

$$3 = \alpha_1 + \alpha_2 \qquad 1 = 2\alpha_2 - 3\alpha_1$$
$$9 = 3\alpha_1 + 3\alpha_2 \qquad 10 = 5\alpha_2 + 0\alpha_1 \Rightarrow$$
$$\alpha_2 = 2$$
$$\Rightarrow \alpha_1 = 1.$$

So $a_n = 1 \cdot (-3)^n + 2 \cdot (2)^n$.

(b) Solve the recurrence

$$a_n = 3a_{n-1} - 4a_{n-3};\ a_0 = 2, a_1 = 0, a_2 = 10.$$

[*Use are allowed to use the fact that* $x^3 - 3x^2 + 4 = (x-2)^2(x+1)$.]

Using the same $a_n = r^n$ sub. as in part(a), we get the charateristic poly.

$$r^3 - 3r^2 + 4 = 0$$

by hint we have $(r-2)^2(r+1) = 0$

So $r_1 = 2$ (twice)

$r_2 = -1$ once.

using the repeated root solution we have

$$a_n = (\alpha_1 + n\alpha_2)(2)^n + \alpha_3(-1)^n.$$

using $a_0 = 2$, $\alpha_1 = 0$, $\alpha_2 = 10$ we sub in to get

① $2 = \alpha_1 + \alpha_3$., ② $0 = 2\alpha_1 + 2\alpha_2 - \alpha_3$.

③ $10 = 4\alpha_1 + 8\alpha_2 + \alpha_3$.

①+③ gives $10 = 6\alpha_1 + 10\alpha_2$

①+② gives. $2 = 3\alpha_1 + 2\alpha_2$

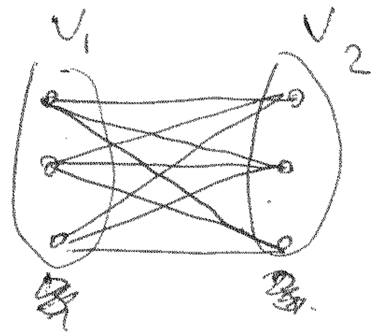
13. (20 points)

(a) Suppose we have a graph $G = (V, E)$. Define what it means for the graph G to be bipartite.

A graph G is bipartite if its vertex set V can be split into two sets V_1 & V_2 s.t. $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$ (called a disj. partition). Where every edge in the graph connects a vertex in V_1 to a vertex in V_2 (i.e. no (V_1 to V_1) or (V_2 to V_2) edges).

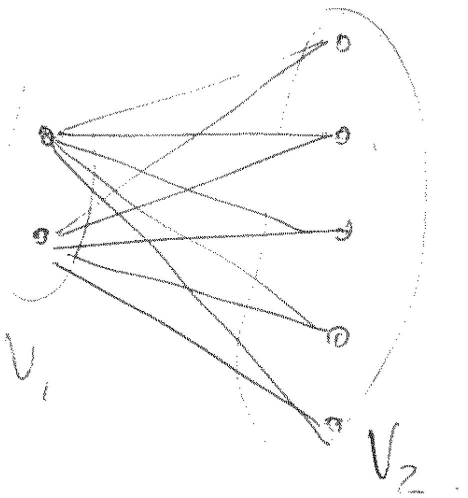
(b) Draw the complete bipartite graph $K_{n,m}$ for

(i) $n = m = 3$.



9 edges

(ii) $n = 2$, $m = 5$.



10 edges.

(c) How many edges does the graph $K_{n,m}$ have in this general case.

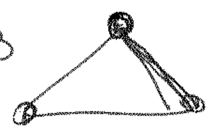
$K_{n,m}$ has $n \cdot m$ edges.

(as each of the n points in V_1 is connected to m pts in V_2.)

(d) Which of the following graphs are bipartite. *If yes make the two distinct sets of vertices clear, if no no further explanation is required.*

(i) K_3.

K_3

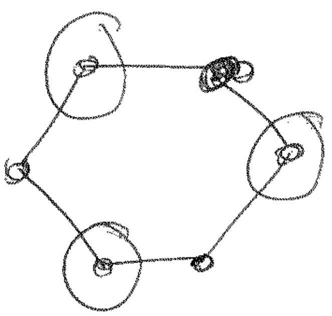


Not bipartite.

(ii) C_6.

Yes.

C_6 is bipartite



Circled vertices in V_1

Non-circled in V_2.

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