## Part-A

**1. (20 pts)**

[10 points](a) Determine if

$$[(\neg p) \wedge ((p \vee r) \wedge q)] \rightarrow (q \vee r)$$

is a tautology using truth tables. **Show all your work.**

| $p$ | $q$ | $r$ | $p \vee r$ | $(p \vee r) \wedge q$ | $(\neg p) \wedge ((p \vee r) \wedge q) = Hyp$ | $q \vee r = Concl$ | $Hyp \rightarrow Concl$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | T | T |
| T | T | F | T | T | F | T | T |
| T | F | T | T | F | F | T | T |
| T | F | F | T | F | F | F | T |
| F | T | T | T | T | T | T | T |
| F | T | F | F | F | F | T | T |
| F | F | T | T | F | F | T | T |
| F | F | F | F | F | F | F | T |

Always T. Tautology!

(b) Determine the truth value of each of these statements if the domain of each variable consists of all real numbers. **(Circle your answers. Explain your answers briefly.)**

[2 points](i) $\forall x \exists y (x^3 = y)$ is T/F

Any real number $x$ can cube it to get $x^3 = y$ another real number.

[2 points](ii) $\forall x \exists y (x = y^2)$ is T/F

$x = -1$, $\nexists y \in \mathbb{R}$ s.t. $y^2 = -1$

[2 points](iii) $\exists x \forall y\ (xy = 0)$ is T/F

$x = 0$

[2 points](iv) $\forall x (x \neq 0 \rightarrow \exists y (xy = 1))$ is T/F

$y = \frac{1}{x}$

[2 points](v) $\exists x \exists y (x + y \neq y + x)$ is T/F

$x + y = y + x$ always

**2. (20 pts)**

[5 points](a) Use the Euclidean Algorithm to find the greatest common divisor (gcd) of 2000 and 6080. **Show all your work.**

$(6080) = 3(2000) + 80$ ← gcd = last nonzero remainder

$(2000) = 25(80) + 0$

[4 points](b) We describe a coding method used in Agency X. First we convert letters into numbers via

A=0,B=1,C=2,D=3,E=4,F=5,G=6,H=7,I=8,J=9,K=10,L=11,M=12,N=13,O=14, P=15, Q=16,R=17,S=18,T=19,U=20,V=21,W=22,X=23,Y=24,Z=25.

Then the agency codes these via the function

$$f(x) = 7x + 9 \text{ (modulo 26)}.$$

Thus C is coded as 23 since $7(2) + 9 = 23$ and H is coded as 6 since $7(7) + 9 = 58 \equiv 6$ modulo 26.

As an agent of Agency X, you receive a coded letter to signal your next action. The coded number is 11. Use that 15 is the inverse of 7 modulo 26 to **find the original letter** by solving

$$11 = 7x + 9 \text{ (modulo 26)}.$$

$11 - 9 = 7x \quad (\text{mod } 26)$

$2 = 7x \quad (\text{"})$

$(15)(2) = \underbrace{(15)(7)}_{\equiv 1}x \quad (\text{"})$

1 as 15 is mult inverse of 7

$30 = 1x \quad \text{mod } 26 \longrightarrow x = 4 \text{ mod } 26$

letter = E

[6 points](c) Use the Euclidean algorithm to find the multiplicative inverse of 23 modulo 31. That is find the integer $s$ such that $23s \equiv 1$ (modulo 31). (Use the canonical representative of $s$ modulo 31, i.e., find $s$ in the range 0-30. **You must use the Euclidean algorithm method to get full credit.**)

$(31) = 1(23) + 8$
$(23) = 2(8) + 7$
$(8) = 1(7) + 1$ = gcd
$(7) = 7(1) + 0$

$\therefore\ 1 = 1(8) - 1(7)$
$= 1(8) - 1((23) - 2(8)) = 3(8) - 1(23)$
$1 = 3((31) - 1(23)) - 1(23) = 3(31) - 4(23)$
$1 = 3(31) - 4(23)$
So $1 \equiv (-4)(23) \bmod 31$

$\therefore\ s = -4 \bmod 31$
$= -4 + 31$ "
$\boxed{s = 27 \bmod 31}$

[5 points](d) Find the canonical representative of $2^{20680}$ modulo 5 and modulo 11.

FLT: $2^{p-1} \equiv 1 \bmod p$ $p$ prime $\neq 2$
$\therefore\ 2^4 \equiv 1 \bmod 5$ and $2^{10} \equiv 1 \bmod 11$

$\therefore\ 2^{20680} = 2^{(5170)(4)+0} \equiv (2^4)^{5170} \cdot 2^0 \bmod 5$
$\equiv 1^{5170} \cdot 1$ "
$\equiv \boxed{1} \bmod 5$

$2^{20680} = 2^{(2068)(10)+0} \equiv (2^{10})^{2068} \cdot 2^0 \bmod 11$
$\equiv 1^{2068} \cdot 1$ "
$= \boxed{1} \bmod 11$

**3. (20 pts)**

[10 points](a) **Prove** that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}=\frac{n}{2n+1} \Big\} P(n)$$

for any positive integer $n$.

$P(1)$: $\frac{1}{1\cdot 3} \stackrel{\checkmark}{=} \frac{1}{2(1)+1}$

Now will show $P(n) \to P(n+1)$. So have $P(n)$ is T.
Start with left hand side of $P(n+1)$:

$$\underbrace{\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}}+\frac{1}{(2n+1)(2n+3)} \xrightarrow{\text{Using } P(n)} \frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)}$$

$$=\frac{n(2n+3)+1}{(2n+1)(2n+3)}$$

$$=\frac{2n^2+3n+1}{(2n+1)(2n+3)}$$

$$=\frac{\cancel{(2n+1)}(n+1)}{\cancel{(2n+1)}(2n+3)}$$

$$=\frac{n+1}{2n+3}$$

So $P(n+1)$ is T ✓

[10 points](b)The Ackermann function is defined using the rule

$$A(m,n)=\begin{cases} 2n & \text{if } m=0\\ 0 & \text{if } m\geq 1 \text{ and } n=0\\ 2 & \text{if } m\geq 1 \text{ and } n=1\\ A(m-1,A(m,n-1)) & \text{if } m\geq 1 \text{ and } n\geq 2.\end{cases}$$

for all positive integers $n$.

**Find** $A(2,1)$ **and** $A(2,2)$.

$A(2,1)=\boxed{2}$ (case 3 above)

$A(2,2)=A(1,A(2,1))$ (case 4 above)

$=A(1,2)$

$=A(0,A(1,1))$ (case 4 above)

$=2A(1,1)$ (case 1 above)

$=2(2)$ (case 3 above)

$=\boxed{4}$

**4. (20 pts)** [12 points](a) Consider the system of congruences:

$$\begin{cases} x & \equiv a_1 \pmod{3} \leftarrow m_1 \\ x & \equiv a_2 \pmod{4} \leftarrow m_2 \\ x & \equiv a_3 \pmod{5} \leftarrow m_3 \end{cases}$$

Use the Chinese Remainder Theorem formula to find explicit integers $A, B, C$ such that

$$x = Aa_1 + Ba_2 + Ca_3 \pmod{60}$$

where $A, B, C$ should be explicitly calculated integers. (**Show your work**). In particular use your formula to solve for the smallest positive integer $x$ satisfying the specific system of congruences:

$$\begin{cases} x & \equiv 1 \pmod{3} \\ x & \equiv 3 \pmod{4} \\ x & \equiv 2 \pmod{5} \end{cases}$$

$M = m_1 m_2 m_3 = (3)(4)(5) = 60$
$M_1 = m_2 m_3 = (4)(5) = 20$
$M_2 = m_1 m_3 = (3)(5) = 15$
$M_3 = m_1 m_2 = (3)(4) = 12$

| | | |
|---|---|---|
| $M_1 y_1 \equiv 1 \bmod m_1$ | $M_2 y_2 \equiv 1 \bmod m_2$ | $M_3 y_3 \equiv 1 \bmod m_3$ |
| $20 y_1 \equiv 1 \bmod 3$ | $15 y_2 \equiv 1 \bmod 4$ | $12 y_3 \equiv 1 \bmod 5$ |
| $2 y_1 \equiv 1 \bmod 3$ | $3 y_2 \equiv 1 \bmod 4$ | $2 y_3 \equiv 1 \bmod 5$ |
| $\rightarrow y_1 = 2$ | $\rightarrow y_2 = 3$ | $\rightarrow y_3 = 3$ |

As any mod-eqn can have other equivalent solutions also.

$x = M_1 y_1 a_1 + M_2 y_2 a_2 + M_3 y_3 a_3 = (20)(2)a_1 + (15)(3)a_2 + (12)(3)a_3$

$\boxed{x = 40a_1 + 45a_2 + 36a_3 \bmod 60}$

$x = 40(1) + 45(3) + 36(2)$
$= 40 + 135 + 72$
$= 247 \bmod 60$
$x = \boxed{7} \bmod 60$

[8 points](b) Let $P(n)$ be the statement that $n$ cents of postage can be made exactly by using just 4-cent and 5-cent stamps. Determine the T/F value of $P(1)$ through $P(15)$. Determine the minimum $N$ such that $P(n)$ is true for all $n \geq N$ and prove your answer using induction.

$P(1)$ ✗   $P(9)$ ✓ $4+5$
$P(2)$ ✗   $P(10)$ ✓ $2\times5$
$P(3)$ ✗   $P(11)$ ✗
$P(4)$ ✓ $4$   $P(12)$ ✓ $3\times4$
$P(5)$ ✓ $5$   $P(13)$ ✓ $2\times4+1\times5$
$P(6)$ ✗   $P(14)$ ✓ $2\times5+1\times4$
$P(7)$ ✗   $P(15)$ ✓ $3\times5$
$P(8)$ ✓ $2\times4$

$\boxed{N=12}$

$P(n) \longrightarrow P(n+4)$ as can add a 4¢ stamp!

Thus

$$\left.\begin{array}{l} P(12) \rightarrow P(12+4k) \\ P(13) \rightarrow P(13+4k) \\ P(14) \rightarrow P(14+4k) \\ P(15) \rightarrow P(15+4k) \end{array}\right\} k \geq 0$$

Any $n \geq 12$ is congruent mod 4 to one of 12, 13, 14, 15 so is captured. Thus $P(n)$ T for all $n \geq 12$.

**5. (20 pts)** (a) For the purpose of this problem, simple functions will mean any of the functions $1, n, n^2, n^3, \ldots, b^n, n!, log(n), n^n$ and any of their products like $n^2 log(n)$ or $n!2^n$. ($b$ is any base with $b > 1$.)

Give a big-O estimate for each of the following functions. For the function $g$ in your estimate "$f(n)$ is $O(g(n))$", use a simple function $g$ of **smallest** order. **Show enough work so that it can be understood how you arrived at your answer.**

[3 points](i) $(n^3 + n^2 + 1)(3n^2 + 5) + 4n^4 log(n)$

$O(n^3)$ $O(n^2)$ $O(n^4 \log n)$

$O(n^5)$

$\boxed{O(n^5)}$

[3 points](ii) $n^3 log(n) + n^5 + 2^n$

largest is $2^n$

so $\boxed{O(2^n)}$

[3 points](iii) $\frac{n^4+2n^3+3n+1}{n^2+5}$ $= \frac{O(n^4)}{\Theta(n^2)} = \boxed{O(n^2)}$

[3 points](iv) $1^5 + 2^5 + 3^5 + \cdots + n^5$ $= \boxed{O(n^6)}$

[8 points](b) Write pseudocode for an algorithm that takes as input a list of $n$ integers and finds the number of positive integers in the list.

```
procedure POS(a1, a2, .., an ; ints)
  counter := 0
  For i = 1 to n
    If ai > 0 then counter := counter + 1
Output(counter)
```

**Part-B**

**1. (25 pts)** [5 points](a) Find the coefficient of $x^3$ in the expansion of $(2x-1)^{20}$ using the binomial theorem.

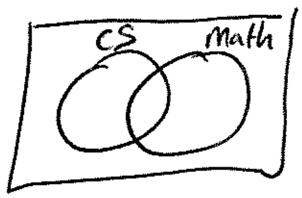
$$\text{Term with Ans} = C(20,3)(2x)^3(-1)^{17}$$

$$= \underbrace{\boxed{C(20,3)\cdot 2^3\cdot(-1)^{17}}}_{\text{coefficient}}\, x^3$$

$$\text{Ans} = \boxed{\frac{20\cdot19\cdot18}{3\cdot2\cdot1}\cdot 2^3\cdot(-1)^{17}}$$

[5 points](b) Every student in a discrete math class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class total if there are 34 computer science majors (including joint majors), 21 math majors (including joint majors) and 7 joint majors in the class?

$$|\text{Class}| = |CS \cup \text{Math}|$$
$$= |CS| + |\text{Math}| - |CS \cap \text{math}|$$
$$= 34 + 21 - 7$$
$$= \boxed{48}$$




[5 points](c) From a group of 6 men and 8 women a committee consisting of 4 men and 3 women is to be formed. How many different committees are possible if 2 of the men refuse to serve together?

# of possible subcomms without 2 man restriction $= C(6,4)\cdot C(8,3)$

# possible subcomms with these particular 2 men $= \overbrace{C(4,2)}^{\text{choose 2 more other men}}\cdot\overbrace{C(8,3)}^{\text{choose 3 women}}$

$\therefore$ Ans
$= C(6,4)C(8,3)$
$\quad - C(4,2)C(8,3)$
$= \boxed{[C(6,4)-C(4,2)]\cdot C(8,3)}$

[5 points](d) What is the minimum number of people needed to guarantee that at least 7 people have the same birthmonth?

Min $N$ s.t. $\lceil \frac{N}{12} \rceil = 7$

So solve $\frac{N}{12} = 6 \rightarrow N = 72 \xrightarrow{\text{Add } 1} \boxed{N=73}$

[5 points](e) How many positive integers between 100 and 999 inclusive are divisible by 5 or by 4 (inclusive or)?

Div5: $\underset{9ch}{\_}\ \underset{10ch}{\_}\ \underset{2ch\ (0 \text{ or } 5)}{\_} = \boxed{180}$

Div4: $100 = 4(25), \ldots, 1000 = 4(250)$

So multiples of 4 in range are $25 \times 4$ thru $249 \times 4$ inclusive.

So #Div4 in range = # of numbers in $\{25, \ldots, 249\}$

$= 249 - 24$

$= \boxed{225}$

Div 4 & 5 = Div 20

$100 = 5(20)$, $1000 = 50(20)$

#Div20 in range = #$\{5, \ldots, 49\}$

$= 49 - 4 = \boxed{45}$

Ans = Div5 + Div4 − Div20

$= 180 + 225 - 45$

$= \boxed{360}$

**2. (25 pts)** (a)Bill the truck driver pays tolls with only pennies (1 cent) and nickels (5 cents). He pays the toll by throwing one coin at a time at the toll collector. (When a human collects the tolls, this makes them unhappy.)

Let $T(n)$ be the number of ways that Bill can pay a toll of $n$ cents (where the order that Bill throws the pennies and nickels matters).

Then $T(1) = 1, T(2) = 1$ and $T(6) = 3$ since Bill can pay 6 cents either by using all pennies, throwing a nickel followed by a penny or throwing a penny followed by a nickel.

[5 points](i) What are $T(3), T(4), T(5)$?

$T(3) = 1$ all pennies
$T(4) = 1$ "
$T(5) = 2$ all pennies or nickel

[7 points](ii) Find a recurrence for $T(n)$ when $n > 5$.

$T(n)$: last coin — nickel, previously pay $n-5$ in $T(n-5)$ ways
— penny, " " $n-1$ in $T(n-1)$ ways

$\therefore$ $\boxed{T(n) = T(n-1) + T(n-5)}$

[13 points](b) Solve the recurrence $a_n = 5a_{n-1} - 6a_{n-2}$ subject to the initial conditions $a_0 = 1$ and $a_1 = 0$.

$$r^n = 5r^{n-1} - 6r^{n-2}$$

$\div r^{n-2}$

$\boxed{r^2 = 5r - 6 \quad \text{Charpoly}}$

$r^2 - 5r + 6 = 0 \longrightarrow (r-2)(r-3) = 0$

$\longrightarrow r = 2, 3$

$\therefore$ $\boxed{a_n = C_1 2^n + C_2 3^n}$ is general solution

Initial Conditions

$1 = a_0 = C_1 2^0 + C_2 3^0 = C_1 + C_2 \quad C_2 = 1 - C_1$

$0 = a_1 = C_1 2^1 + C_2 3^1 = 2C_1 + 3C_2 \quad 0 = 2C_1 + 3(1 - C_1) \rightarrow C_1 = 3$

$\rightarrow C_2 = -2$

$\therefore$ $\boxed{a_n = 3 \cdot 2^n - 2 \cdot 3^n}$

**3. (25 pts)** (a) Consider the set $V$ consisting of 5 people

$$V = \{Ann, Bob, Chuck, Dave, Elaine\}.$$

Let $G$ be the graph with vertices given by $V$ and where we join two vertices by an edge if the corresponding people are mutual friends.
(We will not consider anyone to be mutual friends with themself so there will be no loops in this graph.)
Suppose we know:

- Bob is mutual friends with exactly 2 of the others but not with Elaine.
- Ann is mutual friends with everyone (all 4 of the others).
- Dave is mutual friends with exactly 1 of the others.
- Chuck is mutual friends with exactly 3 of the others.
- Elaine is mutual friends with exactly N of the others.

So apriori, we know $N = 0, 1, 2, 3$ or $4$.
[2 points](i)Using information given in the problem, explain why $N = 0$ and $N = 4$ are not possible.

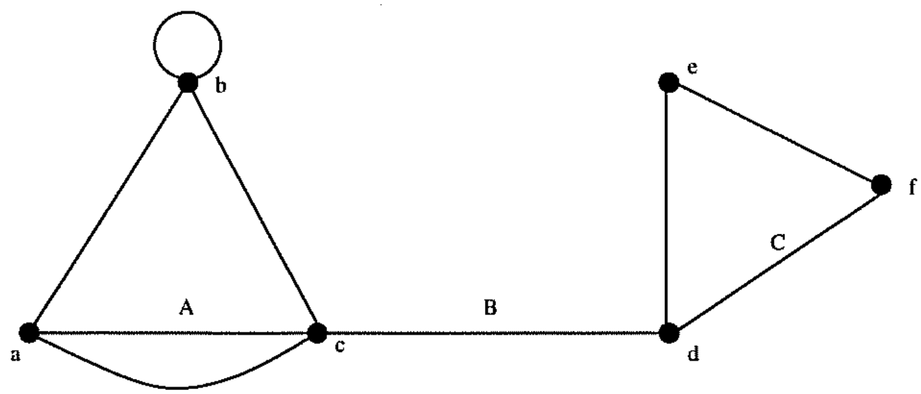
Elaine must be friends with Ann as ~~she~~ Ann is friends with all
So $N > 0$.
Elaine isn't friends with Dave who is only friends with ~~Ann~~ one person which is Ann.
So $N < 4$. (or as Bob not friends with Elaine)

[3 points](ii)Apply the Handshaking theorem to the graph to eliminate two more possibilities and hence determine what $N$ is.

Handshaking: $2+4+1+3+N = 2\#\text{edges} = \text{even}$

$\therefore$ $10+N = \text{even}$
$\therefore$ $N = \text{even}$
so $N \neq 1, 3$
$\longrightarrow$ $\boxed{N=2}$ as $N \neq 0, 4$ by (i)

[5 points](b)Complete the following tables for the graph below.




| Vertex | Degree | Cut Vertex(Y/N) |
|---|---|---|
| a | 3 | N |
| b | 4 | N |
| c | 4 | Y |
| d | 3 | Y |
| e | 2 | N |
| f | 2 | N |

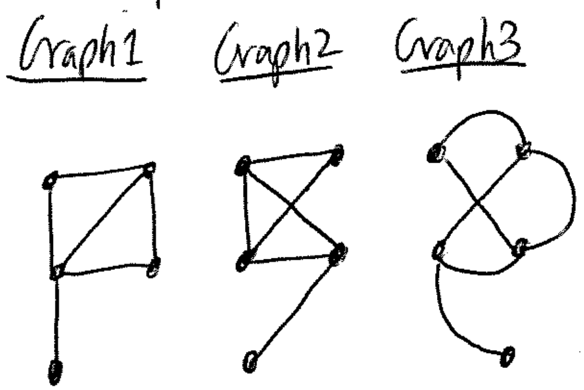
| Edge | Cut Edge(Y/N) |
|---|---|
| A | N |
| B | Y |
| C | N |

[5 points](c) Write down the adjacency matrix for the graph in (b).

$$
\begin{array}{c|cccccc}
 & a & b & c & d & e & f \\
\hline
a & 0 & 1 & 2 & 0 & 0 & 0 \\
b & 1 & 1 & 1 & 0 & 0 & 0 \\
c & 2 & 1 & 0 & 1 & 0 & 0 \\
d & 0 & 0 & 1 & 0 & 1 & 1 \\
e & 0 & 0 & 0 & 1 & 0 & 1 \\
f & 0 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

[5 points](d)Two of the following three graphs are isomorphic and the remaining one is different. Find the odd man out. **Explain how you know.**

Say it was

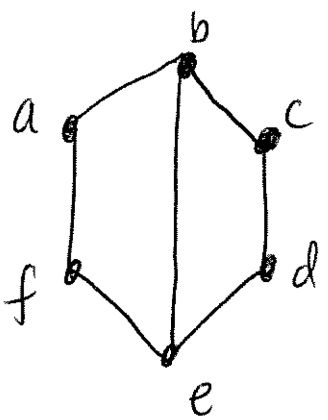



Graph 1 is odd man out

Graph 2 $\cong$ Graph 3

One way to see is deg sequence graph 1 is 4, 3, 2, 2, 1 while graph 2 & 3 it is 3, 3, 3, 2, 1.

[5 points](e) Is the graph below bipartite or not? If it is, provide a bipartite decomposition of the vertices. If not, show some work to indicate how you know it isn't.




thrown in $V_1$

$V_1 = \{a, c, e\}$

have to be in $V_1$ as b, f in $V_2$

$V_2 = \{b, f, d\}$

have to be in $V_2$ as 'a' in $V_1$

have to be in $V_2$ as c, e in $V_1$

forced. Final check $V_1$, $V_2$ see if have internal connections. As don't this is a bipartite decomposition + graph is bipartite.

**4. (25 pts)** (a) Describe in full detail, the number of vertices, edges, and the degrees for all the vertices in the following graphs:

[3 points](i)$K_n$, the complete graph.

n vertices

$C(n,2) = \binom{n}{2} = \frac{n(n-1)}{2}$ edges

degrees all $n-1$

[3 points](ii)$C_n$, the cycle graph.

n vertices

degree $= 2$ all vertices

$2E = \sum_{v \in V} 2 = 2n \rightarrow E = n$ edges

[3 points](iii)$Q_n$, the hypercube graph.

$2^n$ vertices

degree n all vertices

$2 \times \text{Edges} = \sum_{v \in V} n = n 2^n \rightarrow \#\text{Edges} = n \cdot 2^{n-1}$

[5 points](b)Does $Q_3$ have an Euler path or Euler circuit? If so write one down in either edge or vertex notation, if not explain how you know.
(Recall the vertices in $Q_3$ can be labeled like 010, 110 etc. and you may write a path as 010-110-... etc.)

X Not relevant

[5 points](c)Does $Q_4$ have an Euler circuit? If so write one down in vertex notation, if not explain how you know.
(Recall the vertices in $Q_4$ can be labeled like 0101, 1101 etc. and you may write a path as 0101-1101-... etc.)

X Not relevant

[6 points](d)Define, in as clear a manner as possible, what a Hamilton circuit for a graph is. Do the hypercube graphs $Q_3$ and $Q_4$ have Hamilton circuits? If so write them down in vertex notation.

X Not Relevant.

**Part-C** **This is an Extra Credit Section. It is not worth as many points as Part-A and Part-B of this exam so do not work on it unless you have already worked on those parts. In all of these "historical" problems, any comments might be worth some partial extra credit so if you do try them, say whatever you remember about the problem.**

**1. (3 pts)**

[3 points] How many regions do 7 generic lines in the plane divide the plane into? (Remember in a collection of generic lines, any 2 cross, and no 3 meet at the same point.)

Extra Credit
Find answer
on own!

**2. (3 pts)**

[3 points] 242 people sit in a circle in positions labeled 0 thru 241. Person 0 stabs Person 1, Person 2 stabs Person 3, and they keep going around the circle, the next surviving person stabbing the person after them. Which person is the last person remaining alive?

//

[#3, 4 (Extra credit on planar graphs won't be on our exam)]