MATH 150

FINAL EXAM
20 Dec, 2010

NAME (please print legibly): —————————————————————————

Student ID Number: —————————————————————————

- **No calculators are allowed on this exam.**

- **Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.**

- **You may use a 8" x 11" notesheet (both sides).**

Part-A		
QUESTION	VALUE	SCORE
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

Part-B		
QUESTION	VALUE	SCORE
1	25	
2	25	
3	25	
4	25	
TOTAL	100	

Part-C		
QUESTION	VALUE	SCORE
1	3	
2	3	
3	6	
4	2	
TOTAL	14	

Part-A

1. (20 pts)

[10 points](a) Determine if

$$[(\neg p) \wedge ((p \vee r) \wedge q)] \to (q \vee r)$$

is a tautology using truth tables. **Show all your work.**

(b) Determine the truth value of each of these statements if the domain of each variable consists of all real numbers. **(Circle your answers. Explain your answers briefly.)**

[2 points](i) $\forall x \exists y (x^3 = y)$ is T/F

[2 points](ii) $\forall x \exists y (x = y^2)$ is T/F

[2 points](iii) $\exists x \forall y \ (xy = 0)$ is T/F

[2 points](iv) $\forall x (x \neq 0 \to \exists y (xy = 1))$ is T/F

[2 points](v) $\exists x \exists y (x + y \neq y + x)$ is T/F

2. (20 pts)

[5 points](a) Use the Euclidean Algorithm to find the greatest common divisor (gcd) of 2000 and 6080. **Show all your work.**

[4 points](b) We describe a coding method used in Agency X. First we convert letters into numbers via

A=0,B=1,C=2,D=3,E=4,F=5,G=6,H=7,I=8,J=9,K=10,L=11,M=12,N=13,O=14, P=15, Q=16,R=17,S=18,T=19,U=20,V=21,W=22,X=23,Y=24,Z=25.

Then the agency codes these via the function

$$f(x) = 7x + 9 \text{ (modulo 26)}.$$

Thus C is coded as 23 since $7(2) + 9 = 23$ and H is coded as 6 since $7(7) + 9 = 58 \equiv 6$ modulo 26.

As an agent of Agency X, you receive a coded letter to signal your next action. The coded number is 11. Use that 15 is the inverse of 7 modulo 26 to **find the original letter** by solving

$$11 = 7x + 9 \text{ (modulo 26)}.$$

[6 points](c) Use the Euclidean algorithm to find the multiplicative inverse of 23 modulo 31. That is find the integer s such that $23s \equiv 1$ (modulo 31). (Use the canonical representative of s modulo 31, i.e., find s in the range 0-30. **You must use the Euclidean algorithm method to get full credit.**)

[5 points](d) Find the canonical representative of 2^{20680} modulo 5 and modulo 11.

3. (20 pts)

[10 points](a) **Prove** that

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1) \cdot (2n+1)} = \frac{n}{2n+1}$$

for any positive integer n.

[10 points](b)The Ackermann function is defined using the rule

$$A(m, n) = \begin{cases} 2n & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \text{ and } n = 0 \\ 2 & \text{if } m \geq 1 \text{ and } n = 1 \\ A(m-1, A(m, n-1)) & \text{if } m \geq 1 \text{ and } n \geq 2. \end{cases}$$

for all positive integers n.

   Find $A(2, 1)$ **and** $A(2, 2)$**.**

4. (20 pts) [12 points](a) Consider the system of congruences:

$$\begin{cases} x & \equiv a_1 \ (\text{mod } 3) \\ x & \equiv a_2 \ (\text{mod } 4) \\ x & \equiv a_3 \ (\text{mod } 5) \end{cases}$$

Use the Chinese Remainder Theorem formula to find explicit integers A, B, C such that

$$x = Aa_1 + Ba_2 + Ca_3 \ (\text{mod } 60)$$

where A, B, C should be explicitly calculated integers. (**Show your work**). In particular use your formula to solve for the smallest positive integer x satisfying the specific system of congruences:

$$\begin{cases} x & \equiv 1 \ (\text{mod } 3) \\ x & \equiv 3 \ (\text{mod } 4) \\ x & \equiv 2 \ (\text{mod } 5) \end{cases}$$

[8 points](b) Let $P(n)$ be the statement that n cents of postage can be made exactly by using just 4-cent and 5-cent stamps. Determine the T/F value of $P(1)$ through $P(15)$. Determine the minimum N such that $P(n)$ is true for all $n \geq N$ and prove your answer using induction.

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5. (20 pts) (a) For the purpose of this problem, simple functions will mean any of the functions $1, n, n^2, n^3, \ldots, b^n, n!, log(n), n^n$ and any of their products like $n^2 log(n)$ or $n!2^n$. (b is any base with $b > 1$.)

Give a big-O estimate for each of the following functions. For the function g in your estimate "$f(n)$ is $O(g(n))$", use a simple function g of **smallest** order. **Show enough work so that it can be understood how you arrived at your answer.**

[3 points](i) $(n^3 + n^2 + 1)(3n^2 + 5) + 4n^4 log(n)$

[3 points](ii) $n^3 log(n) + n^5 + 2^n$

[3 points](iii) $\frac{n^4 + 2n^3 + 3n + 1}{n^2 + 5}$

[3 points](iv) $1^5 + 2^5 + 3^5 + \cdots + n^5$

[8 points](b) Write pseudocode for an algorithm that takes as input a list of n integers and finds the number of positive integers in the list.

Part-B

1. (25 pts) [5 points](a) Find the coefficient of x^3 in the expansion of $(2x - 1)^{20}$ using the binomial theorem.

[5 points](b) Every student in a discrete math class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class total if there are 34 computer science majors (including joint majors), 21 math majors (including joint majors) and 7 joint majors in the class?

[5 points](c) From a group of 6 men and 8 women a committee consisting of 4 men and 3 women is to be formed. How many different committees are possible if 2 of the men refuse to serve together?

[5 points](d) What is the minimum number of people needed to guarantee that at least 7 people have the same birthmonth?

[5 points](e) How many positive integers between 100 and 999 inclusive are divisible by 5 or by 4 (inclusive or)?

2. (25 pts) (a)Bill the truck driver pays tolls with only pennies (1 cent) and nickels (5 cents). He pays the toll by throwing one coin at a time at the toll collector. (When a human collects the tolls, this makes them unhappy.)

Let $T(n)$ be the number of ways that Bill can pay a toll of n cents (where the order that Bill throws the pennies and nickels matters).

Then $T(1) = 1, T(2) = 1$ and $T(6) = 3$ since Bill can pay 6 cents either by using all pennies, throwing a nickel followed by a penny or throwing a penny followed by a nickel.

[5 points](i) What are $T(3), T(4), T(5)$?

[7 points](ii) Find a recurrence for $T(n)$ when $n > 5$.

[13 points](b) Solve the recurrence $a_n = 5a_{n-1} - 6a_{n-2}$ subject to the initial conditions $a_0 = 1$ and $a_1 = 0$.

3. (25 pts) (a) Consider the set V consisting of 5 people

$$V = \{Ann, Bob, Chuck, Dave, Elaine\}.$$

Let G be the graph with vertices given by V and where we join two vertices by an edge if the corresponding people are mutual friends.

(We will not consider anyone to be mutual friends with themself so there will be no loops in this graph.)

Suppose we know:

- Bob is mutual friends with exactly 2 of the others but not with Elaine.

- Ann is mutual friends with everyone (all 4 of the others).

- Dave is mutual friends with exactly 1 of the others.

- Chuck is mutual friends with exactly 3 of the others.

- Elaine is mutual friends with exactly N of the others.

So apriori, we know $N = 0, 1, 2, 3$ or 4.

[2 points](i)Using information given in the problem, explain why $N = 0$ and $N = 4$ are not possible.

[3 points](ii)Apply the Handshaking theorem to the graph to eliminate two more possibilities and hence determine what N is.

[5 points](b)Complete the following tables for the graph below.
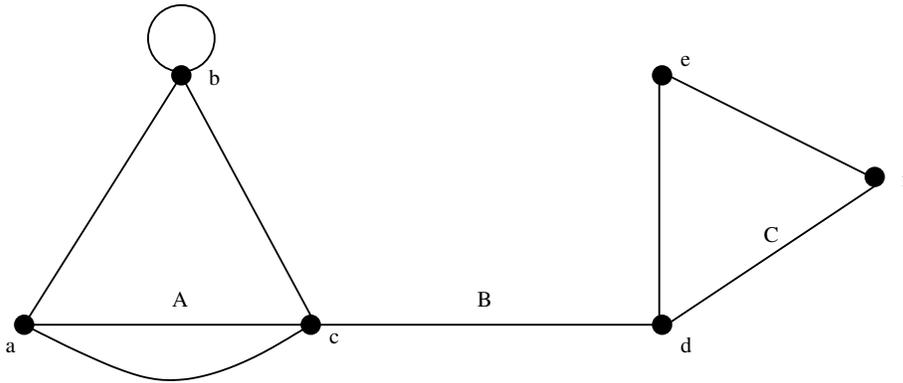
Vertex	Degree	Cut Vertex(Y/N)
a		
b		
c		
d		
e		
f		

Edge	Cut Edge(Y/N)
A	
B	
C	

[5 points](c) Write down the adjacency matrix for the graph in (b).

[5 points](d)Two of the following three graphs are isomorphic and the remaining one is different. Find the odd man out. **Explain how you know.**

[5 points](e) Is the graph below bipartite or not? If it is, provide a bipartite decomposition of the vertices. If not, show some work to indicate how you know it isn't.

4. (25 pts) (a) Describe in full detail, the number of vertices, edges, and the degrees for all the vertices in the following graphs:

[3 points](i)K_n, the complete graph.

[3 points](ii)C_n, the cycle graph.

[3 points](iii)Q_n, the hypercube graph.

[5 points](b)Does Q_3 have an Euler path or Euler circuit? If so write one down in either edge or vertex notation, if not explain how you know.
(Recall the vertices in Q_3 can be labeled like 010, 110 etc. and you may write a path as 010-110-... etc.)

[5 points](c)Does Q_4 have an Euler circuit? If so write one down in vertex notation, if not explain how you know.
(Recall the vertices in Q_4 can be labeled like 0101, 1101 etc. and you may write a path as 0101-1101-... etc.)

[6 points](d)Define, in as clear a manner as possible, what a Hamilton circuit for a graph is. Do the hypercube graphs Q_3 and Q_4 have Hamilton circuits? If so write them down in vertex notation.

Part-C This is an Extra Credit Section. It is not worth as many points as Part-A and Part-B of this exam so do not work on it unless you have already worked on those parts. In all of these "historical" problems, any comments might be worth some partial extra credit so if you do try them, say whatever you remember about the problem.

1. (3 pts)

[3 points] How many regions do 7 generic lines in the plane divide the plane into? (Remember in a collection of generic lines, any 2 cross, and no 3 meet at the same point.)

2. (3 pts)

[3 points] 242 people sit in a circle in positions labeled 0 thru 241. Person 0 stabs Person 1, Person 2 stabs Person 3, and they keep going around the circle, the next surviving person stabbing the person after them. Which person is the last person remaining alive?

3. (6 pts)

[3 points](a) Draw **planar representations** of the complete bipartite graphs $K_{1,n}$ and $K_{2,n}$ as well as for the complete graph K_4. Note **planar representation** means we are looking for a specific type of drawing of these graphs so make sure your drawing satisfies this.

[3 points](b) For which n exactly are the complete graphs K_n planar? For which m, n exactly are the complete bipartite graphs $K_{m,n}$ planar?

4. (2 pts)

[2 points] Every planar map can be colored by _____ colors such that no two adjacent regions (regions that share a boundary curve) share the same color. Your answer should be the least integer that makes the statement correct.