1. (10 pts) Prove the following theorem:

THEOREM: If a, b, c, d are positive integers, and if $a|b$ and $c|d$, then $ac|bd$.

Notes: (i) The notation "$x|y$" means "x divides y".

(ii) Show all steps in your proof (and, as usual, show all work).

2. (10 pts) Is the following statement true or false? If it is True, then supply a detailed PROOF. If it False, then prove that it is False by supplying a COUNTEREXAMPLE. STATEMENT: If a, b and c are positive integers, and if $a|bc$, then either $a|b$ or $a|c$.

3. (15 pts) Prove or disprove the following statement:
STATEMENT: If a, b, m are positive integers, such that

$$a \equiv b (\text{mod } m) \text{ and } c \equiv d \ (\text{mod } m), \text{ then}$$

$$a \cdot c \equiv b \cdot d \ (\text{mod } m) \ .$$

Note: If the statement is TRUE, then supply a complete PROOF. If the statement is FALSE, then show that it is false, by supplying a COUNTEREXAMPLE.

4. (10 pts) Let h be the hashing function $h(k) = k \bmod 101$. (I.e., $h(k) =$ the smallest non-negative integer that is congruent to k modulo 101. This will be one of the integers: $0, 1, \ldots, 100$). Then compute

$$h(104578690)\ .$$

[HINT: We have that $100 \equiv -1 \pmod{101}$. Therefore, $100x \equiv -x \pmod{101}$, for any integer x. Here's how to use this, in computing h (any positive integer): For example, $5762 = 57 \times 100 + 62$. Taking "$x$" above to be "57", $5762 \equiv -57 + 62 = 5 \pmod{101}$ so $h(5762) = 5$.]

5. (10 pts) Consider the linear congruential generator

$$x_{n+1} = (4x_n + 1) \bmod 7$$

with seed $x_0 = 3$.

(I.e., in more mathematical language, consider the sequence $x_n, n \geq 0$, defined by the recursion

$$x_{n+1} = (4x_n + 1) \bmod 7, n \geq 1 ,$$

obeying the initial condition $x_0 = 3$). Then compute *all* of the $x_n, n \geq 0$ explicitly.

[HINT: Start computing $x_0, x_1, x_2, x_3, \ldots$; and pretty soon you'll find it clear what all of them are.]

6. (15 pts) Recall that if n is any positive integer, then $\phi(n)$ is the number of positive integers $\leq n$ that are prime to n. For example, to compute $\phi(8)$, the positive integers ≤ 8 are 1,2,3,4,5,6,7 and 8. Of these, only 1,3,5 and 7 are prime to 8. Therefore, $\phi(8) = 4$. $\phi(n)$ is the Euler phi-function.

(i)(5 pts) Compute $\phi(15)$.

(ii)(10 pts) If p is any prime, compute $\phi(p)$.
[HINT: Try a few primes; the general formula for any prime will soon become clear.]

7. (15 pts)

(i)(3 pts) Convert the hexidecimal expansion $(FAC1E)_{16}$ to its binary expansion.

(ii)(3 pts) Convert the hex number in (i) above into an octal number.

(iii)(3 pts) Convert the hex number in (i) to a decimal number.

(iv)(3 pts) Convert the hex number in (i) to a binary number.

(v)(3 pts) Convert the decimal number $(1326)_{10}$ into a binary number.

Answers:

(i)

(ii)

(iii)

(iv)

(v)

8. (10 pts) We've shown in class that $(*)\ 1 + x + \cdots + x^n = \frac{x^{n+1}-1}{x-1}$, if $x \neq 0$. Use this to show that

$$3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots 3 \cdot 5^n = \frac{3}{4}\left(5^{n+1} - 1\right) ,$$

for any integer $n \geq 0$.

[HINT: Using equation $(*)$ above, you don't have to go through a proof by induction again.]

9. (20 pts) Use induction on the integer n, for $n \geq 0$, to show that

$$1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3} .$$

10. (15 pts) Prove, by induction on n, that

$$2^n > n^2$$

for every integer $n \geq 5$.

11. (15 pts) Recall that an *r-permutation* of a set S is a sequence $(x_1, \ldots, x_r)$ of r different elements of S. E.g., (4,1,2) is a 3-permutation of $\{1, 2, 3, 4, 5\}$; but (4,1,4) is not since the elements in the sequence are not different).

An *r-combination* of a set S is a subset of S having exactly r elements.

(i)(2pts) List all 2-permutations of $\{1, 2, 3\}$.

(ii)(2 pts) List all 3-combinations of $\{1, 2, 3, 4\}$.

(iii)(2 pts) Compute $P(6, 3)$. [Note: You don't have to multiply it out.]

(iv)(2 pts) Compute $C(5, 1)$.

(v)(3 pts) Compute $C(5, 4)$.

(vi)(4 pts) What is the number of all permutations of $\{1, 2, 3, 4\}$? [Recall that a *permutation* of a set S is an n-permutation of S, where $n = |S|$.]

12. (20 pts) Solve the lhrc (linear homogeneous recurrence relation with constant coefficients)

$$a_n = -4a_{n-1} - 4a_{n-2}, n \geq 2$$

obeying the initial conditions

$$a_0 = 1,\ a_1 = 2\ .$$

NOTE: Show all steps and all work.

13. (20 pts) Solve the lhrc

$$a_n = 6a_{n-1} - 9a_{n-2}\ , n \geq 2\ ,$$

obeying the initial conditions

$$a_0 = 2,\ a_1 = -1\ .$$

[Again, show all work.]