MTH150 Midterm Exam 1 Solutions

February 19, 2007

(A1) (a) i. proposition, false
ii. not a proposition, ambiguous (truth depends on the value of n)
iii. not a proposition, not declarative

(b) i. T
ii. F
iii. T
iv. T

(A2) (a) A contingency.

p	q	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$	$[(\neg p) \wedge (\neg q)] \vee p$	$\{[(\neg p) \wedge (\neg q)] \vee p\} \rightarrow q$
T	T	F	F	F	T	T
T	F	F	T	F	T	F
F	T	T	F	F	F	T
F	F	T	T	T	T	F

(b) A contradiction.

p	q	$p \oplus q$	$p \leftrightarrow q$	$(p \oplus q) \leftrightarrow (p \leftrightarrow q)$
T	T	F	T	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

(A3) (a) i. F $(0 > 1)$
ii. T $(4 > 3)$
iii. T ($n = 2$ is an example)
iv. F ($n = 0$ is a counterexample)

(b) i. F $(\pm\sqrt{7} \notin \mathbb{Z})$
ii. T ($n = 0$ is an example)
iii. F ($n = \pm 1$ are two examples)

(c) i. $x = 0$ is a counterexample
ii. $x = -0.5$ is a counterexample
iii. $x = 0$ is a counterexample

(A4) (a) T (given y, $x = y^2$ is an example)
(b) F ($y = -1$ is a counterexample)
(c) T ($x = 0$ is an example)
(d) T ($x = 0$ is an example)
(e) F ($y = 0$ is a counterexample)
(f) T (given $x \neq 0$, $y = 1/x$ is an example)

(g) T (given $x \neq 0$, $y = 1/x$ is the only example)

(h) T ($x = 0$ is an example)

(i) T ($x = y = 0$ is an example)

(j) T (this is the commutative law for addition of real numbers)

(A5) (a) I want to prove the statement "$\forall m, \forall n, ((m \text{ odd}) \wedge (n \text{ odd})) \rightarrow (mn \text{ odd})$" where the domain of each variable consists of all integers. I will use the method of *direct proof.*

1. Let m and n denote arbitrary integers (so I can universally generalize and the end of the proof).
2. Assume m and n are odd. (I'm using the method of direct proof so I need to assert the hypotheses.)
3. Then $m = 2k + 1$ and $n = 2l + 1$ for some integers k and l. (That's the definition of odd.)
4. Then $mn = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l + 1)$. (That's algebra.)
5. So mn is odd. (That's the definition of even.)
5. Thus the implication $((m \text{ odd}) \wedge (n \text{ odd})) \rightarrow (mn \text{ odd})$ is true. (That's what the method of direct proof guarantees us.)
6. Thus $\forall m, \forall n, ((m \text{ odd}) \wedge (n \text{ odd})) \rightarrow (mn \text{ odd})$ is true. (That's universal generalization.)

(b) I want to prove the statement "For all collections of days, if the collection has 36 days, then at least 6 of those days fall on the same day of the week." I will use the method of *proof by contradiciton.* (The method of indirect proof is also applicable.)

1. Consider an arbitrary collection of days (so I can universally generalize and the end of the proof).
2. Assume the collection has 36 days. (I am using the method of proof by contradiction, so I first need to proceed by the method of direct proof.)
3. Assume, contrary to what we want to show, that at most 5 of the days in the collection fall on the same day of the week. (I'm using the method of proof by contradiction, so I have to assume the negation of what I'm trying to prove.)
4. In particular, there are at most 5 Mondays, 5 Tuesdays, 5 Wednesdays, 5 Thursdays, 5 Fridays, 5 Saturdays, and 5 Sundays in my collection. So the number of days in total is at most $(7)(5) = 35$.
5. The previous step contradicts the assumption that there are 36 days.
6. Thus, the assumption that at most 5 of the days in the collection fall on the same day of the week must be wrong. I may assert that at least 6 of any chosen 36 days must fall on the same day of the week. (That's what the method of proof by contradiction guarantees us.)
7. Thus the implication "if the collection has 36 days, then at least 6 of those days fall on the same day of the week" is true. (That's what the method of direct proof guarantees us.)
8. Thus the statement "for all collections of days, if the collection has 36 days, then at least 6 of those days fall on the same day of the week" is true. (That's universal generalization.)

(A6) (a) The statement is true because 5 is a positive integer that equals a third the sum of the positive integers not exceeding it. Indeed, $(1 + 2 + 3 + 4 + 5)/3 = 15/3 = 5$. This proof is *constructive.*

(b) I want to prove the statement "$\forall x, |x| = |-x|$" where the domain of the variable x consists of all real numbers. I will use the method of *proof by cases.*

1. Let x be an arbitrary real number (so I can universally generalize at the end of the proof).
2. Consider three cases.
   i. Suppose $x > 0$. Then $|x| = x$. Also, $-x < 0$ so that $|-x| = -(-x) = x$. In particular $|x| = |-x|$.
   ii. Suppose $x = 0$. Then $|x| = 0$. Also, $-x = 0$ so that $|-x| = 0$. In particular, $|x| = |-x|$.
   iii. Suppose $x < 0$. Then $|x| = -x$. Also, $-x > 0$ so that $|-x| = -x$. In particular, $|x| = |-x|$.

3. The desired result "$|x| = |-x|$" holds in each case and, thus, is true. (That's what the method of proof by cases guarantees us.)
4. Thus the statement "$\forall x, |x| = |-x|$" is true. (That's universal generalization.)

(B1) (a) i. F
ii. T
iii. F
iv. T
v. T

(b) i. 0
ii. 1
iii. 3
iv. 5
v. 8

(B2) (a) $\{0, 1, 2, 3, 4, 5, 6\}$
(b) $\{0, 4\}$
(c) $\{1, 5\}$
(d) $\{2, 3, 6\}$

(B3) (a)

$$\begin{aligned}
x \in \overline{A \cap B} &\Leftrightarrow \neg(x \in A \cap B) && \text{(definition of complement)}\\
&\Leftrightarrow \neg[(x \in A) \wedge (x \in B)] && \text{(definition of intersection)}\\
&\Leftrightarrow [\neg(x \in A)] \vee [\neg(x \in B)] && \text{(DeMorgan's law)}\\
&\Leftrightarrow (x \in \overline{A}) \vee (x \in \overline{B}) && \text{(definition of complement)}\\
&\Leftrightarrow x \in \overline{A} \cup \overline{B} && \text{(definition of union)}
\end{aligned}$$

(b)

$$\begin{aligned}
x \in A \cap (B \cup C) &\Leftrightarrow (x \in A) \wedge (x \in B \cup C) && \text{(definition of intersection)}\\
&\Leftrightarrow (x \in A) \wedge [(x \in B) \vee (x \in C)] && \text{(definition of union)}\\
&\Leftrightarrow [(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)] && \text{(distributive law)}\\
&\Leftrightarrow (x \in A \cap B) \vee (x \in A \cap C) && \text{(definition of intersection)}\\
&\Leftrightarrow x \in (A \cap B) \cup (A \cap C) && \text{(definition of union)}
\end{aligned}$$

(B4) (a) $f(n) = 2n$
(b) $f(n) = \lfloor n/2 \rfloor$
(c) $f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$
(d) $f(n) = 0$

(B5) (a) $7(2)^n$, geometric progression
(b) $7 + 6n$, arithmetic progression

(B6) (a) $999, 999, 999$
(b) 18