

Part A

1. (10 points)

Construct a truth table for each of the following compound propositions and determine whether it is a tautology, contradiction, or contingency.

(a) $\{[(\neg p) \wedge (\neg q)] \vee p\} \rightarrow q$

(b) $(p \oplus q) \leftrightarrow (p \leftrightarrow q)$

2. (10 points)

Determine the truth value of each of the following statements where the domain of each variable consists of all real numbers.

(a) $\forall x, \exists y, y = x^2$

(b) $\forall y, \exists x, y = x^2$

(c) $\exists x, \forall y, xy = 0$

(d) $\forall y, \exists x, xy = 0$

(e) $\forall y, \exists! x, xy = 0$

(f) $\forall x, (x \neq 0) \rightarrow (\exists y, xy = 1)$

(g) $\forall x, (x \neq 0) \rightarrow (\exists! y, xy = 1)$

(h) $\exists x, (x \neq 0) \rightarrow (\forall y, xy = 1)$

(i) $\exists x, \exists y, x + y = y + x$

(j) $\forall x, \forall y, x + y = y + x$

3. (10 points)

(a) Prove that the product of two odd integers is odd. What proof technique are you using?

(b) Prove that at least 6 of any 36 days chosen must fall on the same day of the week. What proof technique are you using?

4. (10 points)

(a) Determine whether each of the following statements is true or false.

(i) $\emptyset \in \emptyset$

(ii) $\emptyset \subseteq \emptyset$

(iii) $\emptyset \subset \emptyset$

(iv) $\emptyset \in \{\emptyset\}$

(v) $\emptyset \subseteq \{\emptyset\}$

(b) Determine the cardinality of each of the following sets.

(i) \emptyset

(ii) $\{\emptyset\}$

(iii) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$

(iv) $\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5\}$

(v) $P(\{1, 2, 3\})$

5. (10 points)

Let $A = \{0, 1, 4, 5\}$ and $B = \{0, 2, 3, 4, 6\}$. Find

(a) $A \cup B$

(b) $A \cap B$

(c) $A - B$

(d) $B - A$

6. (10 points)

(a) Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

(b) Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Part B

1. (10 points)

Use *insertion sort* to sort 6, 2, 3, 1, 5, 4, showing each comparison made and the lists obtained after each iteration of the for loop.

procedure *insertion sort*(a_1, a_2, \dots, a_n : real numbers with $n \geq 2$)

for $j := 2$ **to** n

$i := 1$

while $a_j > a_i$

$i := i + 1$

 Remove a_j and insert at index i (shifting the remaining elements to the right)

{ a_1, \dots, a_n is in increasing order}

2. (10 points)

(a) Evaluate these quantities.

(i) $24 \bmod 5$

(ii) $-97 \bmod 10$

(b) Find the prime factorization of each of these integers.

(i) 257 [Hint: $\sqrt{257} \approx 16.03$.]

(ii) 1027 [Hint: $\sqrt{1027} \approx 32.04$.]

3. (10 points)

(a) Use the extended Euclidean algorithm to express $\gcd(63, 25)$ as a linear combination of 63 and 25.

(b) Solve the following congruence if possible.

$$25x \equiv 6 \pmod{63}$$

If there is a solution, find the unique one satisfying $0 \leq x < 63$. If there is no solution, state so and explain.

4. (10 points)

(a) Conjecture a formula for

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots + \frac{1}{(2n-1)(2n+1)}$$

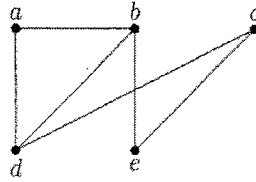
by examining the values of this expression for small values of n .

(b) Prove the formula you conjectured in part (a) using induction.

Part A

1. (10 points)

(a) Represent the following undirected graph by an adjacency matrix.

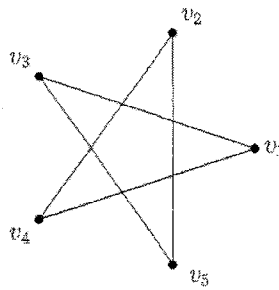
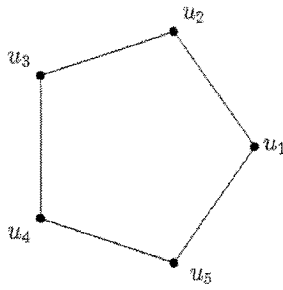


(b) Draw an undirected graph represented by the following adjacency matrix.

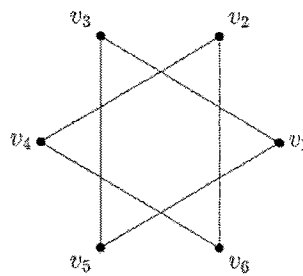
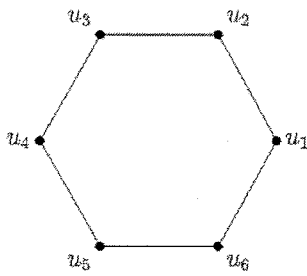
$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$$

2. (10 points)

(a) Find an isomorphism between the following graphs or show that they are not isomorphic.

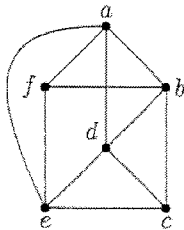


(b) Find an isomorphism between the following graphs or show that they are not isomorphic.

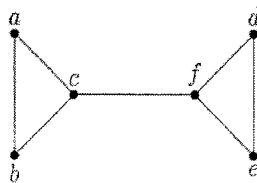


3. (10 points)

- (a) Determine whether the following graph has an Euler circuit. Construct such a circuit if one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists. If no Euler path exists, state so and explain.



- (b) Determine whether the following graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.



4. (10 points)

Consider the following linear homogeneous recurrence relation of degree 2 with constant coefficients.

$$a_n = 7a_{n-1} - 10a_{n-2}$$

(a) Find the general solution.

(b) Find the unique solution satisfying the initial conditions $a_0 = 2$ and $a_1 = 1$.

5. (10 points)

Consider the following linear homogeneous recurrence relation of degree 2 with constant coefficients.

$$a_n = -6a_{n-1} - 9a_{n-2}$$

(a) Find the general solution.

(b) Find the unique solution satisfying the initial conditions $a_0 = 3$ and $a_1 = -3$.