Math 150: Discrete Math

Midterm 1

Oct 10, 2019

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Instructions:

1. Indicate your instructor with a check in the appropriate box:

Zhang	MW 9:00	
Lorman	MW 10:25	
Mkrtchyan	MW 12:30	
Lubkin	MW 3:25	

2. Read the notes below:

   - **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
   - **Notes of any kind are strictly forbidden.**
   - **Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
   - **You are responsible for checking that this exam has all 9 pages.**

3. Read the following Academic Honesty Statement and sign:

   I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature:______________________________

1. (20 points)

Let p and q be the propositions:

$$p: \text{ You get a speeding ticket.}$$

$$q: \text{ You drive over 80 miles per hour.}$$

Write these propositions using p and q and logical connectives (including negations).

(a) If you do not drive over 80 miles per hour, then you will not get a speeding ticket.

Answer: $\neg q \to \neg p$

(b) You drive over 80 miles per hour, but you do not get a speeding ticket.

Answer: $q \wedge \neg p$

(c) You will get a speeding ticket if you drive over 80 miles per hour.

Answer: $q \to p$

(d) Whenever you get a speeding ticket, you are driving over 80 miles per hour.

Answer: $p \to q$

2. (20 points)

(a) Show that $\neg p \to (q \to r)$ and $q \to (p \lor r)$ are logically equivalent.

p	q	r	$q \to r$	$\neg p \to (q \to r)$	$p \lor r$	$q \to (p \lor r)$
T	T	T	T	T	T	T
T	T	F	F	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	T	T	T	T
F	T	F	F	F	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	T

Since the last column matches the fifth column, the statements are logically equivalent.

(b) Show that $((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$ is a tautology.

p	q	r	$p \lor q$	$\neg p \lor r$	$q \lor r$	$(p \lor q) \land (\neg p \lor r)$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	F	T
F	F	F	F	T	F	F	T

Since the last column is all true, $((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$ is a tautology.

3. (20 points)

(a) Prove that for all integers n, n is even if and only if $n^2 + 5$ is odd.

Suppose n is even. Then $\exists k \in \mathbb{Z}$ such that $n = 2k$. Then $n^2+5 = 4k^2+5 = 2(2k^2+2)+1$. Since k is an integer, so is $2k^2 + 1$. Since $n^2 + 5$ is twice an integer plus one, it is odd.

Need to show if $n^2 + 5$ is odd then n is even. Suppose n is odd. Then $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$. Then $n^2 + 5 = 4k^2 + 4k + 1 + 5 = 2(2k^2 + 2k + 3)$. Since $k \in \mathbb{Z}$, we have $2k^2 + 2k + 3 \in \mathbb{Z}$, so $n^2 + 5$ is twice an integer, so it's even. That's a contradiction, so n cannot be odd, so must be even.

(b) Prove or disprove the following statement: the sum of any two irrational numbers is irrational.

This is not true. For example let $x = \sqrt{2}$ and $y = -\sqrt{2}$. We know from class that x is irrational. y is also irrational, since if it were rational, x, which is $-y$ would also be rational. So, x, y are both irrational, but $x + y = 0$ is rational.

4. (20 points) Let A, B and C be sets. Prove or give a counter example. If proving that two sets are equal, you should show that an arbitrary element from the first set is in the second set and that an arbitrary element from the second set is in the first set. Simply giving a membership table will earn very little credit, if any.

(a) $(A \cup B) - A = B - (A \cap B)$

Suppose $x \in (A \cup B) - A$. Then $x \in A \cup B$ and $x \notin A$. $x \in A \cup B$ means $x \in A$ or $x \in B$. But we know $x \notin A$ so we must have $x \in B$. On the other hand $x \notin A$ implies $x \notin A \cap B$. Since $x \in B$ and $x \notin A \cap B$ we have $x \in B - (A \cap B)$. Since this holds for every $x \in (A \cup B) - A$, we have $(A \cup B) - A \subset B - (A \cap B)$.

Now, suppose $x \in B - (A \cap B)$. Then $x \in B$ and $x \notin A \cap B$. It follows that $x \not inA$. Now, $x \in B$ implies $x \in A \cup B$. Combined with $x \notin A$ gives $x \in (A \cup B) - A$. Since this holds for all $x \in B - (A \cap B)$, we have $B - (A \cap B) \subset (A \cup B) - A$.

Combining the two parts we obtain $B - (A \cap B) = (A \cup B) - A$

(b) $A - (B - C) = (A - B) \cup C$

This is not true. For example let A and B be empty, but $C = \{1\}$. Then $A - (B - C)$ is empty but $(A - B) \cup C = \{1\}$.

Of course there are many examples. One way to come up with examples is to draw a Venn diagram or an inclusion table.

5. (20 points)

(a) Rewrite each statement below as a logically equivalent statement, with all implications replaced by a combination of logical connectives ($\neg$, $\wedge$, $\vee$), and such that all negation symbols appear immediately before a predicate (that is, not before a quantifier, a logical connective or a parenthesis).

(i) $\neg\forall y\exists x(P(x,y)\vee Q(x,y))$

Answer: $\exists y\forall x(\neg P(x,y)\wedge\neg Q(x,y))$

(ii) $\neg(\exists x\exists y\neg P(x,y)\wedge\forall x\forall yQ(x,y))$

Answer: $(\forall x\forall yP(x,y))\vee(\exists x\exists y\neg Q(x,y))$

(b) Consider the two statements:

"$\forall x(P(x) \rightarrow Q(x))$" and "$\forall x P(x) \rightarrow \forall x Q(x)$". Either prove that the two statements are logically equivalent, or else disprove this assertion, by finding a counterexample. If the latter, determine whether either statement implies the other.

Counterexample: Idea - Let the domain, $P(x)$ and $Q(x)$ be such that for some x from the domain $P(x)$ is true and $Q(x)$ is false, and for some other x, $P(x)$ is false. Since there are x's such that $P(x)$ is false, $\forall x P(x)$ is false so $\forall x P(x) \rightarrow \forall x Q(x)$ is true. However, since there is an x such that $P(x)$ is true but $Q(x)$ is false, $\forall x(P(x) \rightarrow Q(x))$ is false. As a concrete example, take for example the domain $D = \{0, 1\}$, $P(x)$ to be $x > 0$ and $Q(x)$ to be $x > 2$.

Let's show "$\forall x(P(x) \rightarrow Q(x))$" implies "$\forall x P(x) \rightarrow \forall x Q(x)$"

Suppose "$\forall x(P(x) \rightarrow Q(x))$" is true. Now, if $\forall x P(x)$ is false, then "$\forall x P(x) \rightarrow \forall x Q(x)$" is true. If $\forall x P(x)$ is true, then from $\forall x(P(x) \rightarrow Q(x))$ we obtain that $\forall x Q(x)$ is also true, so "$\forall x P(x) \rightarrow \forall x Q(x)$" is also true. Thus, in all cases, if "$\forall x(P(x) \rightarrow Q(x))$" is true, then "$\forall x P(x) \rightarrow \forall x Q(x)$" is also true.