Math 150: Discrete Mathematics

Exam 2-Solutions

Thursday, March 28, 2024

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Indicate your instructor with a check in the appropriate box:

Dannenberg	MW 10:25-11:40am	
Kumar	TR 9:40-10:55am	

- You are responsible for checking that this exam has all 10 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers, unless specified otherwise.

Please **COPY** the HONOR PLEDGE and **SIGN:**

I affirm that I will not give or receive any unauthorized help on this exam,
and all work will be my own.

HONOR PLEDGE:

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YOUR SIGNATURE:______________________________

1. (20 points)

(a) Consider the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined as

$$f(x) = \frac{1}{4x} + 3.$$

(i) Is f injective (one-to-one)? Prove or show why not?

f is injective.

Suppose $x_1, x_2 \in \mathbb{R} - \{0\}$ and $f(x_1) = f(x_2)$. This means that

$$\frac{1}{4x_1} + 3 = \frac{1}{4x_2} + 3 \iff \frac{1}{4x_1} = \frac{1}{4x_2} \iff \frac{1}{x_1} = \frac{1}{x_2} \iff x_1 = x_2.$$

Therefore, f is injective.

(ii) Is f surjective (onto)? Prove or show why not?

The function f is not surjective because there exists an element $y = 3 \in \mathbb{R}$ for which

$$f(x) = \frac{1}{4x} + 3 \neq 3$$

for every $x \in \mathbb{R} - \{0\}$.

(iii) Is f bijective? Justify your answer.

No, f is not bijective. To be bijective, it would need to be both injective and surjective, but by part (ii) it is not surjective.

(b) Consider the function $f : \mathbb{R} - \{0\} \to \mathbb{R} - \{3\}$ defined as

$$f(x) = \frac{1}{4x} + 3.$$

Is f surjective (onto)? Prove or show why not?

To show that f is surjective, take an arbitrary $y \in \mathbb{R} - \{3\}$. We seek an $x \in \mathbb{R} - \{0\}$ for which $f(x) = y$, that is for which

$$\frac{1}{4x} + 3 = y.$$

Solving for x yields

$$x = \frac{1}{4(y-3)},$$

which is defined because $y \neq 3$. Thus, for any $y \in \mathbb{R} - \{3\}$, we have

$$f\left(\frac{1}{4(y-3)}\right) = y.$$

Therefore, f is surjective.

2. (16 points)

(a) **(8 points)** Consider $f(n) = 1^{11} + 2^{11} + 3^{11} + 4^{11} + \ldots + n^{11}$. Let a be the least integer such that f is big - $\mathcal{O}$ of n^a. What is a? Prove that f is big - $\mathcal{O}$ of n^a. State the values used for witnesses C and k.

Show your work clearly so that it can be understood how you arrived at your answer.

$a = 12$.

For each $1 \leq j \leq n$; $j^{11} \leq n^{11}$. So for all $n \geq 1$,

$$\begin{aligned} &1^{11} + 2^{11} + 3^{11} + 4^{11} + \ldots + n^{11} \\ \leq\ &n^{11} + n^{11} + n^{11} + \ldots + n^{11} \\ \leq\ &n \cdot n^{11} \\ =\ &n^{12}. \end{aligned}$$

Hence, $f(n) = 1^{11} + 2^{11} + 3^{11} + 4^{11} + \ldots + n^{11}$ is big - $\mathcal{O}$ of n^{12} with witnesses $C = 1$ and $k = 1$.

(b) **(8 points)**

(i) Define what it means for f to be big - Ω of g. Your definition must include the constants C and k.

We say that f is big - Ω of g if there are positive constants C and k such that

$$|f(x)| \geq C|g(x)|$$

whenever $x > k$.

(ii) Determine whether $f(x) = x \log x$ is big - Ω of x^2. If not, then give a brief explanation. If yes, provide the value used for witnesses C and k.

$f(x) = x \log x$ is NOT big - Ω of x^2.

If $x \log x$ is big - Ω of x^2, there exists positive constants C and k such that

$$x \log x \geq Cx^2, \quad \forall x > k \iff \log x \geq Cx, \quad \forall x > k,$$

which has NO solutions since $\log x$ grows more slowly that x. Hence, $f(x) = x \log x$ is not big - Ω of x^2.

3. (12 points) Mark the following statements as True or False. If a statement is false, find a counterexample that shows why it is false. (If it is true, you do not need to explain why.) All the variables represent integers.

(a) If $x \equiv 4 \pmod 6$ and $y \equiv 2 \pmod 6$, then $x + y \equiv 0 \pmod 6$.

This is TRUE since $x + y \equiv 4 + 2 \pmod 6 \equiv 6 \pmod 6 \equiv 0 \pmod 6$

(b) If $x \equiv 5 \pmod 6$ and $y \equiv -2 \pmod 6$, then $x \cdot y \equiv 2 \pmod 6$.

This is also TRUE since $-10 \equiv 2 \pmod 6$.

(c) If $x \equiv 2 \pmod 6$ and $y \equiv 2 \pmod 6$, then $\frac{x}{y} \equiv 1 \pmod 6$.

This is FALSE. Division only works if the numbers we are dividing are relatively prime to the modulus. In this case, we can take, for example, $x = 8$ and $y = 2$. Then $\frac{x}{y} = 4$, and $4 \not\equiv 2 \pmod 6$.

(d) $a \equiv b \pmod m$ if and only if $a \textbf{ mod } m = b \textbf{ mod } m$.

This is TRUE (theorem from Textbook in Section 4.1)

4. (16 points) Consider the binary number 101011001110. Find the hexadicmal expansion by:

(a) converting from binary to decimal and decimal to hexadecimal.

Binary to decimal:

$$\begin{aligned}(101011001110)_2 &= 2^{11} + 2^9 + 2^7 + 2^6 + 2^3 + 2^2 + 2^1 \\ &= 2048 + 512 + 128 + 64 + 8 + 4 + 2 \\ &= 2766.\end{aligned}$$

Decimal to hexadecimal:

$$\begin{aligned}2766 &= 16 \cdot 172 + 14 \longrightarrow a_0 = 14 \equiv E \\ 172 &= 16 \cdot 10 + 12 \longrightarrow a_1 = 12 \equiv C \\ 10 &= 16 \cdot 0 + 10 \longrightarrow a_2 = 10 \equiv A.\end{aligned}$$

So

$$(101011001110)_2 = (2766)_{10} = (ACE)_{16}.$$

(b) converting directly from binary to hexadecimal.

Group into blocks of four (adding zeros to the left most block if necessary):

$$\begin{aligned}(1010)_2 &= 2^3 + 2^1 = 10 = (A)_{16} \\ (1100)_2 &= 2^3 + 2^2 = 12 = (C)_{16} \\ (1110)_2 &= 2^3 + 2^2 + 2^1 = 14 = (E)_{16}.\end{aligned}$$

Hence,

$$(101011001110)_2 = (ACE)_{16}.$$

5. (20 points) Use modular exponentiation to find 6^{90} **mod** 13. Your answer should be an integer between 0 and 12, inclusive.

Write

$$90 = 64 + 16 + 8 + 2 = 2^6 + 2^4 + 2^3 + 2^1,$$

and then compute

$$\begin{aligned}
6^{2^0} \textbf{ mod } 13 &= 6 \textbf{ mod } 13 = 6 \\
6^{2^1} \textbf{ mod } 13 &= 36 \textbf{ mod } 13 = 10 \textbf{ mod } 13 = \mathbf{-3} \\
6^{2^2} \textbf{ mod } 13 &= 100 \textbf{ mod } 13 = 9 \\
6^{2^3} \textbf{ mod } 13 &= 81 \textbf{ mod } 13 = \mathbf{3} \\
6^{2^4} \textbf{ mod } 13 &= 9 \;\; \textbf{ mod } 13 = \mathbf{-4} \\
6^{2^5} \textbf{ mod } 13 &= 81 \textbf{ mod } 13 = 3 \\
6^{2^6} \textbf{ mod } 13 &= 9 \textbf{ mod } 13 = \mathbf{-4}.
\end{aligned}$$

Multiplying the four terms in bold, we get

$$\begin{aligned}
6^{90} \textbf{ mod } 13 &= (-3)(3)(-4)(-4) \textbf{ mod } 13 \\
&= (12)(-12) \textbf{ mod } 13 \\
&= (12)(1) \textbf{ mod } 13 \qquad\qquad \text{since} \quad -12 = 13 \cdot (-1) + 1 \\
&= 12 \textbf{ mod } 13 = 12.
\end{aligned}$$

6. (16 points) Find Bezout coefficients for 32 and 246. Show your complete work clearly.
Hint: Use the extended Euclidean algorithm.

We first use Euclidean algorithm to find the $\gcd(32, 246)$:

$$\begin{aligned} 246 &= 32 \cdot 7 + 22 \\ 32 &= 22 \cdot 1 + 10 \\ 22 &= 10 \cdot 2 + 2 \\ 10 &= 2 \cdot 5 + 0 \end{aligned}$$

Thus, $\gcd(32, 246) = 2$. Using the above, we have

$$22 = 246 - 32 \cdot 7, \quad 10 = 32 - 22 \cdot 1,$$

and

$$\begin{aligned} 2 &= 22 - 10 \cdot 2 \\ &= 22 - (32 - 22 \cdot 1) \cdot 2 \\ &= 22 \cdot 3 - 32 \cdot 2 \\ &= (246 - 32 \cdot 7) \cdot 3 - 32 \cdot 2 \\ &= 246 \cdot 3 - 32 \cdot 23. \end{aligned}$$

Hence, Bezout coefficients for 246 and 32 are 3 and -23.