Math 150: Discrete Mathematics

Midterm Exam 2 - Practice Exam B - Solutions

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Indicate your instructor with a check in the appropriate box:

Dannenberg	MW 10:25-11:40am	
Kumar	TR 9:40-10:55am	

- You are responsible for checking that this exam has all 12 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers, unless specified otherwise.

Please **COPY** the HONOR PLEDGE and **SIGN:**

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

HONOR PLEDGE:

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YOUR SIGNATURE:______________________________

1. (16 points)

(a) Find the prime factorization for 630.

$$630 = 2(315) = (2)(5)(63) = (2)(5)(3)(21) = (2)(5)(3)(3)(7) = 2^1\, 3^2\, 5^1\, 7^1.$$

(b) Find the binary and hexadecimal representation for the number with decimal representation 151.

$$\begin{aligned}
151 &= (2)(75) + 1 \rightarrow a_0 = 1 \\
75 &= (2)(37) + 1 \rightarrow a_1 = 1 \\
37 &= (2)(18) + 1 \rightarrow a_2 = 1 \\
18 &= (2)(9) + 0 \rightarrow a_3 = 0 \\
9 &= (2)(4) + 1 \rightarrow a_4 = 1 \\
4 &= (2)(2) + 0 \rightarrow a_5 = 0 \\
2 &= (2)(1) + 0 \rightarrow a_6 = 0 \\
1 &= (2)(0) + 1 \rightarrow a_7 = 1
\end{aligned}$$

So $151 = (10010111)_2$. To change from binary to hexadecimal, group the binary digits in groups of 4 and convert to single hexadecimal digit. Thus,

$$151 = (10010111)_2 = (97)_{16}.$$

(c) Find $\gcd(5040, 1000)$.

$$\begin{aligned} 5040 &= 1000 \cdot 5 + 40 \\ 1000 &= 40 \cdot 25 + 0 \end{aligned}$$

Hence, $\gcd(5040, 1000) = 40$.

(d)

$$B(m,n) = \begin{cases} m + nB(m+1, n-1) & \text{if } \ n > 1 \\ 2m \ \text{ if } \ n = 1 \end{cases}$$

Find $B(3,3)$.

$$B(3,3) = 3 + 3B(4,2) = 3 + 3(4 + 2B(5,1)) = 15 + 6(2 \cdot 5) = 15 + 60 = 75.$$

2. (16 points) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ that is

(a) one-to-one but not onto.

$$f(n) = 2n.$$

(b) onto but not one-to-one.

$$f(n) = \lfloor n/2 \rfloor$$

(c) both onto and one-to-one (but different from the identity function).

$$f(n) = \begin{cases} n+1 & \text{if n is even} \\ n-1 & \text{if n is odd} \end{cases}$$

(d) neither one-to-one nor onto.

$$f(n) = 0.$$

3. (15 points) Use the bubble sorting algorithm to order the following integers:

$$11, 5, 3, 7, 10, 1, 4, 9, 2.$$

Clearly show the steps of this algorithm for each pass.

1st sweep: (guarantees that the largest element is in correct position)

11	5	5	5	5	5	5	5	5
5	11	3	3	3	3	3	3	3
3	3	11	7	7	7	7	7	7
7	7	7	11	10	10	10	10	10
10 $\longrightarrow$	10 $\longrightarrow$	10 $\longrightarrow$	10 $\longrightarrow$	11 $\longrightarrow$	1 $\longrightarrow$	1 $\longrightarrow$	1 $=$	1
1	1	1	1	1	11	4	4	4
4	4	4	4	4	4	11	9	9
9	9	9	9	9	9	9	11	2
2	2	2	2	2	2	2	2	**11**

2nd sweep: (guarantees that the two largest elements are in correct position)

5	3	3	3	3	3	3	3
3	5	5	5	5	5	5	5
7	7	7	7	7	7	7	7
10	10	10	10	1	1	1	1
1 $\longrightarrow$	1 $\longrightarrow$	1 $\longrightarrow$	1 $\longrightarrow$	10 $\longrightarrow$	4 $\longrightarrow$	4 $=$	4
4	4	4	4	4	10	9	9
9	9	9	9	9	9	10	2
2	2	2	2	2	2	2	**10**
11	**11**	**11**	**11**	**11**	**11**	**11**	**11**

3rd sweep:

3	3	3	3	3	3	3	
5	5	5	5	5	5	5	
7	7	7	1	1	1	1	
1	1	1	7	4	4	4	
4 $\longrightarrow$	4 $\longrightarrow$	4 $\longrightarrow$	4 $\longrightarrow$	7 $\longrightarrow$	7 $=$	7	
9	9	9	9	9	9	2	
2	2	2	2	2	2	**9**	(guarantees that the
10	**10**	**10**	**10**	**10**	**10**	**10**	three largest elements
11	**11**	**11**	**11**	**11**	**11**	**11**	are in correct position)

4th sweep:

3	3	3	3	3	3	
5	5	1	1	1	1	
1	1	5	4	4	4	
4	4	4	5	5	5	
7 $\longrightarrow$	7 $\longrightarrow$	7 $\longrightarrow$	7 $\longrightarrow$	7 $=$	2	
2	2	2	2	2	**7**	(guarantees that
9	**9**	**9**	**9**	**9**	**9**	the four
10	**10**	**10**	**10**	**10**	**10**	largest elements
11	**11**	**11**	**11**	**11**	**11**	are in correct position)

5th sweep:

3	1	1	1	1	
1	3	3	3	3	
4	4	4	4	4	
5	5	5	5	2	
2 $\longrightarrow$	2 $\longrightarrow$	2 $\longrightarrow$	2 $=$	**5**	(guarantees
7	**7**	**7**	**7**	**7**	that the
9	**9**	**9**	**9**	**9**	five largest
10	**10**	**10**	**10**	**10**	elements are
11	**11**	**11**	**11**	**11**	in correct position)

6th sweep:

1	1	1	1	
3	3	3	3	
4	4	4	2	
2	2	2	**4**	(guarantees
5 $\longrightarrow$	**5** $\longrightarrow$	**5** $=$	**5**	that the
7	**7**	**7**	**7**	six largest
9	**9**	**9**	**9**	elements
10	**10**	**10**	**10**	are in
11	**11**	**11**	**11**	correct position

7th sweep:

1		1		1	
3		3		2	
2		2		**3**	(guarantess that
4		**4**		**4**	the seven
5	$\longrightarrow$	**5**	=	**5**	largest
7		**7**		**7**	elements
9		**9**		**9**	are in
10		**10**		**10**	correct
11		**11**		**11**	position)

In the 8th sweep, we have one comparison since there are two elements to compare, which are in correct order so nothing happens, thereby completing the bubble sort.

8th sweep:

1		1
2		**2**
3		**3**
4		**4**
5	=	**5**
7		**7**
9		**9**
10		**10**
11		**11**

4. (12 points)

(a) Consider

$$f(x) = \frac{x^3 + x^2 + 1}{x^2 + 1}.$$

Let n be the least integer such that f is big - $\mathcal{O}$ of x^n. What is n? Prove that f is big - $\mathcal{O}$ of x^n for the n you found. State the values used for C and k.

$n = 1$. For $x \geq 1$,

$$\frac{x^3 + x^2 + 1}{x^2 + 1} \leq \frac{x^3 + x^2}{x^2} = x + 1 \leq 2x.$$

Hence, f is big - $\mathcal{O}$ of x with witnesses $C = 2$ and $k = 1$.

(b) Show that x^3 is not big - $\mathcal{O}$ of x^2.

Proceed by contradiction. Assume there exist C and k such that $x^3 \leq Cx^2$ for all $x > k$. Thus, if $x \neq 0$, we have $x \leq C$ for all $x > k$. However, this fails for large values of x such as $x = |C| + |k| + 1$.

5. (12 points)

(a) What is $-97 \textbf{ mod } 11$.

We have

$$-97 \textbf{ mod } 11 = (-9 \cdot 11 + 2) \textbf{ mod } 11 = 2.$$

(b) Is 193 a prime number? If yes, show that it is prime. If not, find the prime factorization of it.

Yes. Observe that $13 < \sqrt{193} < 14$. List the primes not exceeding $\sqrt{193}$: $2, 3, 5, 7, 11, 13$. Check each prime until you find a divisor: $193 = 96 \cdot 2 + 1$, $193 = 64 \cdot 3 + 1$, $193 = 38 \cdot 5 + 3$, $193 = 27 \cdot 7 + 4$, $193 = 17 \cdot 11 + 6$, $193 = 14 \cdot 13 + 11$. There are no prime divisors, so 193 is prime.

6. (15 points)

(a) Suppose that a, b and c are non-zero integers and that $(ac)\big|(bc)$. Then is it true that necessarily $a\big|b$? Either prove that the assertion is true, or else construct an explicit counterexample.

Yes, it is true. Let $a, b, c \in \mathbb{Z}$, all non-zero. Suppose $(ac)\big|(bc)$. Then there exists $k \in \mathbb{Z}$ such that $bc = ack$. Since $c \neq 0$, we may divide by c to obtain $b = ak$. It follows that $a\big|b$.

(b) Suppose that a, b and c are non-zero integers and that $a \big| (bc)$ and that $a \nmid b$. Then is it true that necessarily $a \big| c$? Either prove that the assertion is true, or else construct an explicit counterexample.

No! Counterexample: $a = 6$, $b = 2$, $c = 3$. We have $bc = 6$ and so $a \big| (bc)$ but $a \nmid b$ (also $a \nmid c$). Must have a to be relatively prime to b and c for the above to be true.

7. (14 points)

(a) Use the Euclidean Algorithm to find $\gcd(111, 201)$, showing all of your steps.

$$\begin{aligned} 201 &= 111 \cdot 1 + 90 \\ 111 &= 90 \cdot 1 + 21 \\ 90 &= 21 \cdot 4 + 6 \\ 21 &= 6 \cdot 3 + 3 \\ 6 &= 3 \cdot 2 + 0 \end{aligned}$$

Hence, $\gcd(111, 201) = 3$.

(b) The following is the Euclidean Algorithm applied to the integers 29 and 12:

$$\begin{aligned} (29) &= 2(12) + 5 \\ (12) &= 2(5) + 2 \\ (5) &= 2(2) + 1 \\ (2) &= 2(1) + 0 \end{aligned}$$

Run the algorithm "backwards" to write $\gcd(12, 29)$ as a linear combination of 12 and 29. Show your work.

Form the given Euclidean algorithm, we see that $\gcd(12, 29) = 1$. Thus,

$$\begin{aligned} 1 &= 1(5) - 2(2) \\ &= 1(5) - 2(1(12) - 2(5)) = 5(5) - 2(12) \\ &= 5(1(29) - 2(12)) - 2(12) \\ &= 5(29) - 12(12). \end{aligned}$$