Math 150: Discrete Mathematics

Midterm Exam 2 - Practice Exam A - Solutions

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Indicate your instructor with a check in the appropriate box:

Dannenberg	MW 10:25-11:40am	
Kumar	TR 9:40-10:55am	

- You are responsible for checking that this exam has all 9 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers, unless specified otherwise.

Please **COPY** the HONOR PLEDGE and **SIGN:**

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

HONOR PLEDGE:

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YOUR SIGNATURE:______________________________

1. (20 points)

(a) Write 99 in base 2.

$$
\begin{aligned}
99 &= 2 \cdot 49 + 1 \rightarrow a_0 = 1 \\
49 &= 2 \cdot 24 + 1 \rightarrow a_1 = 1 \\
24 &= 2 \cdot 12 + 0 \rightarrow a_2 = 0 \\
12 &= 2 \cdot 6 + 0 \rightarrow a_3 = 0 \\
6 &= 2 \cdot 3 + 0 \rightarrow a_4 = 0 \\
3 &= 2 \cdot 1 + 1 \rightarrow a_5 = 1 \\
1 &= 2 \cdot 0 + 1 \rightarrow a_6 = 1
\end{aligned}
$$

So $99 = (1100011)_2$.

(b) Write 7798 in hexadecimal.

$$
\begin{aligned}
7798 &= 16 \cdot 487 + 6 \rightarrow a_0 = 6 \\
487 &= 16 \cdot 30 + 7 \rightarrow a_1 = 7 \\
30 &= 16 \cdot 1 + 14 \rightarrow a_2 = 14 \equiv E \\
1 &= 16 \cdot 0 + 1 \rightarrow a_4 = 1
\end{aligned}
$$

So $7798 = (1E76)_{16}$.

(c) Write 3 in base 7.

$$3 = 7 \cdot 0 + 3 \rightarrow a_0 = 3. \text{ So } \ 3 = (3)_7.$$

(d) Find $(2AE01)_{16} + (AA1)_{16}$, giving your answer in base 16.

$$
\begin{array}{r}
1 \\
2\ A\ E\ 0\ 1 \\
A\ A\ 1 \\
------ \\
2\ B\ 8\ A\ 2
\end{array}
$$

$E = 14$ and $A = 10$. So $E + A = 24$ in base 10. Thus, $24 \cdot 16^2 = (16 + 8) \cdot 16^2 = 1 \cdot 16^3 + 8 \cdot 16^2$. Hence, sum $= 8$ and carry $= 1$. Now $A + 1 = 11 = B$

(e) Find $(222)_3 \times (28)_9$, giving your answer in base 3 or base 9.

$(223)_3 = 2 \cdot 3^2 + 2 \cdot 3 + 2 = 26$ and $(28)_9 = 2 \cdot 9 + 8 = 26$. Thus,

$$(222)_3 \times (28)_9 = 26^2 = (20 + 6)^2 = 400 + 240 + 36 = 676$$

in base 10. Now

$$
\begin{aligned}
676 &= 9 \cdot 75 + 1 \to a_0 = 1 \\
75 &= 9 \cdot 8 + 3 \to a_1 = 3 \\
8 &= 9 \cdot 0 + 8 \to a_2 = 8
\end{aligned}
$$

So $676 = (831)_9$.

2. (15 points) Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(k) = k^3$ for all $k \in \mathbb{Z}$.

(a) Is f injective? Prove or show why not.

f is injective.

For $x \in \mathbb{R}$, let $g(x) = x^3$, so that f is the restriction of g from $\mathbb{R}$ to $\mathbb{Z}$. Then $g : \mathbb{R} \to \mathbb{R}$ is differentiable with $g'(x) > 0$, for all $x \neq 0$, so g is strictly increasing and hence injective: if $x \neq y$, then WLOG can assume $x < y$ and then $x^3 < y^3$, so that $x^3 \neq y^3$. Hence $g : \mathbb{R} \to \mathbb{R}$ is injective, from which it follows that $f : \mathbb{Z} \to \mathbb{Z}$ is injective.

Alternatively, factor $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$; since the second factor is > 0 for all $x, y \neq 0$ (as can be seen by completing the square), $(\forall x, y \in \mathbb{R}) x \neq y \to x^3 \neq y^3$.

(b) Is f surjective? Prove or show why not.

No, f is not surjective. For example, 2 is not in the range of f, since $2^{1/3}$ is not rational, much less an integer. Alternatively, $1^3 = 1$, $2^3 = 8$, and by the strictly increasing behavior of f, there is no integer n with $1 < n^3 < 8$.

(c) Is f bijective? Prove or show why not.

No, f is not bijective. To be bijective, it would need to be both injective and surjective, but by part (b) it is not surjective.

3. (15 points)

(a) What is 43 **mod** 21.

Since $43 = 21 \cdot 2 + 1$, we have that $43 \equiv 1 \pmod{21}$.

(b) Find 43^{230} **mod** 21.

From part (a), we know that $43 \equiv 1 \pmod{21}$. Therefore,

$$43^{230} \ \mathbf{mod}\ 21 \equiv 1^{230} \bmod 21 \equiv 1 \pmod{21} = 1.$$

(c) Show that if $\gcd(x, p) = 1$, then there is an integer y such that $xy \equiv 1 \pmod{p}$.

Using Bezout's Theorem since $\gcd(x, p) = 1$, there exists integers s, t such that

$$xs + pt = 1,$$

i.e., $pt = 1 - xs$. So $p \,\big|\, (1 - xs)$, which yields that

$$1 - xs \equiv 0 \pmod{p} \iff 1 \equiv xs \pmod{p}.$$

So let $s = y$ and we have

$$xy \equiv 1 \pmod{p},$$

as desired.

4. (20 points)

(a) Define what it means for f to be big - Θ of g. Your answer should feature four constants C_1, C_2 and k_1, k_2.

We have that there exists C_1, k_1 such that $|f(x)| \leq C_1|g(x)|$ for all $x \geq k_1$, and there exists C_2, k_2 such that $|f(x)| \geq C_2|g(x)|$ for all $x \geq k_2$. If both of the above conditions hold, we say that f is big - Θ of g.

(b) Show that $x^2 - 6x + 7$ is big - Ω of x. State the values you use for C and k.

Note that for $x \geq 7$, we have $x^2 \geq 7x$. So $x^2 - 6x + 7 \geq 7x - 6x + 7 = x + 7 \geq x$. Hence, $x^2 - 6x + 7$ is big - Ω of x with witnesses $C = 1$ and $k = 7$.

(c) Show that $\log_{10}(x)$ is big - Θ of $\log_2(x)$.

First, observe that $\log_{10}(x) = \log_{10}(2) \times \log_2(x)$. Since $\log_{10}(x)$ and $\log_2(x)$ only differ by a (positive) constant, there are of the same order. Hence, $\log_{10}(x)$ is big - Θ of $\log_2(x)$.

(d) Show that $f(n) = n^3$ is not big - Ω of $g(n) = 3n^3 \log n$.

Suppose that $f(n) = n^3$ is big - Ω of $g(n) = 3n^3 \log n$. Then there exists constants C, k such that $n^3 \geq Cn^3 \log(n)$, for all $n > k$. Dividing by $n^3 > 0$, this is equivalent with $1 \geq C \log(n)$, for all $n > k$, which is equivalent with $\frac{1}{C} \geq \log(n)$, for all $n > k$. However, we know that $\log(n) \to \infty$ as $n \to \infty$, and thus is not bounded above by any constant. **Contradiction**.

5. (14 points) Prove that if a and b are positive integers such that $\text{lcm}(a, b) = ab$, then a and b are relatively prime.

We know from a Theorem discussed in lectures that

$$\gcd(a, b) \cdot \text{lcm}(a, b) = ab,$$

thus

$$\gcd(a, b) = \frac{ab}{\text{lcm}(a, b)}.$$

Hence, if $\text{lcm}(a, b) = ab$, then

$$\gcd(a, b) = \frac{ab}{ab} = 1,$$

which is the definition of a and b being relatively prime.

6. (16 points)

(a) Find $\gcd(159, 509)$ using the Euclidean Algorithm, showing all of your steps.

$$\begin{aligned} 509 &= 159 \cdot 3 + 32 \\ 159 &= 32 \cdot 4 + 31 \\ 32 &= 31 \cdot 1 + 1 \\ 31 &= 1 \cdot 31 + 0 \end{aligned}$$

Hence, $\gcd(159, 509) = 1$.

(b) Write $\gcd(159, 509)$ as a linear combination of 159 and 509 with integer coefficients. Show your work.

Using the above, we have

$$\begin{aligned} 1 &= 32 - 31 \\ &= 32 - (159 - 32 \cdot 4) = 32 \cdot 5 - 159 \\ &= (509 - 159 \cdot 3) \cdot 5 - 159 \\ &= 509 \cdot 5 - 159 \cdot 16. \end{aligned}$$