Math 150: Discrete Mathematics

Midterm Exam 1

Thursday, October 3, 2024

NAME (please print legibly): ~~M~~ Solutions

Your University ID Number: ____________________

Your University email ____________________

Indicate your instructor with a check in the appropriate box:

Dannenberg	MW 2:00-3:15pm	
Dannenberg	MW 12:30-1:45pm	
Almomani	TR 2:00-3:15pm	
Nathan	MW 4:50-6:05pm	
Dannenberg	**Math 150A**	

- You are responsible for checking that this exam has all 7 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

____

____

____

YOUR SIGNATURE: ____________________

1. (20 points) Determine the truth value of each of the following propositions if the universe of discourse for all variables is the set of integers $\mathbb{Z}$. **For this problem, you do not need to justify your answers.**

1. $\forall m(m^2 - 1 \geq 0)$

$m = 0$

$\boxed{F}$

2. $\forall m \exists n(n \cdot m = n + m)$

$\hookrightarrow$ $nm = n+m$

$\hookrightarrow$ $n(m-1) = m$

$\hookrightarrow$ $n = \frac{m}{m-1} \Rightarrow$ not an integer if $m=3$

$\boxed{F}$

3. $\exists m \forall n((n^2 \geq m) \wedge (n \leq m))$

False, just pick $n > m$

$\boxed{F}$

4. $\exists m \exists n\,[(n^m = 16) \longrightarrow (3m - n + 1 = 5)]$

Let $n = 1$, $m = 0$

$\boxed{T}$

2. (20 points) Prove or disprove the following statement.
If x and y are non-zero integers such that $x \cdot y$ is even and $5x - 3y$ is odd, then either x or y is odd. **In this problem, you may only use the definitions of even and odd numbers.**

We proceed by contrapositive.
The contrapositive of the statement is:
For $x, y \in \mathbb{Z} \setminus \{0\}$, if x and y are even, then xy is odd or $5x-3y$ is even.

Assume x, y are both even. Then $x = 2k$ for some $k \in \mathbb{Z}$ and $y = 2j$ for some $j \in \mathbb{Z}$.
Then $5x - 3y = 10k - 6j = 2(5k - 3j)$.
As $5k-3j$ is an integer, we have $5x-3y$ is even.
Thus, the statement: "xy is odd or $5x-3y$ is even" is True, proving our implication.

3. (20 points)

In this problem, consider the logical statement

$$((\neg p \vee q) \longrightarrow p) \equiv p.$$

1. Justify this statement using a truth table.

p	q	$\neg p \vee q$	$(\neg p \vee q) \to p$
F	F	T	F
F	T	T	F
T	F	F	T
T	T	T	T

Note that the columns corresponding to p and $(\neg p \vee q) \to p$ are identical, thus these two statements are logically equivalent.

2. Justify this statement using logical equivalences. **You do not need to write the names of each equivalence you use, but your process should be clear from your work.**

$(\neg p \vee q) \to p \equiv \neg(\neg p \vee q) \vee p$ (conditional disjunction equivalence)

$\equiv (p \wedge \neg q) \vee p$ (De Morgan's Law)

$\equiv \cancel{(p \vee p) \wedge (p \vee \neg q)}$ ~~(Distrib)~~

p (Absorption Law)

4. (15 points) Prove or disprove the following identity for sets A, B, C

$$\overbrace{(A \setminus B) \cup (A \setminus C)}^{E} = \overbrace{A \cap \overline{B \cap C}}^{F}.$$

Let's do double inclusion.

($E \subseteq F$) Let $x \in (A \setminus B) \cup (A \setminus C)$.

Then $x \in A \setminus B$ or $x \in A \setminus C$, so

$(x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$

Thus, $x \in A$ ~~or~~ and $(x \notin B$ or $x \notin C)$

so $x \in A$ and $\neg(x \in B$ and $x \in C)$

so $x \in A$ and $x \notin B \cap C$

so $x \in A \cap \overline{B \cap C}$.

($F \subseteq E$) Let $x \in A \cap \overline{B \cap C}$. Then $x \in A$ and $x \notin \overline{B \cap C}$. Accordingly, $x \notin B$ or $x \notin C$ – it cannot be in both. Thus $\underbrace{x \in A \text{ but not } B}_{x \in A \setminus B}$ or $\underbrace{x \in A \text{ but not } C}_{x \in A \setminus C}$

Therefore $x \in (A \setminus B) \cup (A \setminus C)$.

5. (25 points)

1. For sets A, B give a logical definition of $A \subseteq B$.

$A \subseteq B$ iff for all $x \in A$, $x \in B$

– In other words, $\forall x (x \in A \rightarrow x \in B)$

2. Let A, B be sets. Show that the following are equivalent. **You need to reprove any claims, you cannot just apply the homework.**

   (a) $A \subseteq B$

   (b) $\mathscr{P}(A) \subseteq \mathscr{P}(B)$

   (c) For any set C, if $C \subseteq A$ then $C \subseteq B$.

((a) → (b)) Suppose $A \subseteq B$, and $E \in P(A)$. Then $E \subseteq A$.
Suppose $x \in E$. Then $x \in A$, since $E \subseteq A$, so $x \in B$ since $A \subseteq B$.
Thus $E \subseteq B$, so $E \in P(B)$. Since E was arbitrary, $P(A) \subseteq P(B)$.

((b) → (c)) Suppose $P(A) \subseteq P(B)$. This means $\forall C (C \in P(A) \rightarrow C \in P(B))$

By definition of the power set, this means
$\forall C (C \subseteq A \rightarrow C \subseteq B)$
which is exactly what we wanted to prove.

((c) → (a)) Suppose that for any C, if $C \subseteq A$ then $C \subseteq B$.
Since $A \subseteq A$, we conclude that $A \subseteq B$.

■

This page intentionally left blank.