Math 150: Discrete Mathematics

Midterm Exam 1

Tuesday, February 20, 2024 - *v3*

NAME (please print legibly): Solutions

Your University ID Number: ______

Your University email ______

Indicate your instructor with a check in the appropriate box:

Dannenberg	MW 10:25-11:40am	
Kumar	TR 9:40-10:55am	

- You are responsible for checking that this exam has all 8 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers, unless specified otherwise.

Please **COPY** the HONOR PLEDGE and **SIGN:**

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

HONOR PLEDGE:

______

______

______

YOUR SIGNATURE: ______

1. (15 points) Let p, q, r be propositions. Use truth table to show that $(\neg p) \to (q \to r)$ is logically equivalent to $q \to (p \vee r)$. **You must explain what about your table shows that the two statements are logically equivalent.**

p	q	r	$\neg p$	$q \to r$	$(\neg p) \to (q \to r)$	$p \vee r$	$q \to (p \vee r)$
T	T	T	F	T	T	T	T
T	T	F	F	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	T

Columns corresponding to $(\neg p) \to (q \to r)$ and $q \to (p \vee r)$ have same truth values, thus logically equivalent.

2. (20 points) The domain for all the variables below is the set of integers, $\mathbb{Z}$. Determine the truth value of each of the following propositions. **For this problem, you do not need to justify your answers.**

(a) $(\forall m)\,(m^2 \geq 1)$

FALSE; Take $m = 0$.

(b) $(\exists m)(\forall n)\,(n \cdot m = n + m)$

FALSE; negation is $\forall m \exists n\,(n \cdot m \neq n + m)$

if $m = 0$, take $n = 1$

if $m \neq 0$, take $n = 0$

(c) $(\forall m)(\exists n)\,((n^2 > m) \wedge (n \leq m))$

TRUE; if $m \geq 0$, take $n = -(m+1)$

if $m < 0$, take $n = m - 1$

(d) $(\exists m)(\exists n)\,[(n^m = 9) \rightarrow (n - 2m = 5)]$

TRUE; for $n = m = 1$, $n^m = 9$ is F,

so conditional is T,

regardless of the truth

value of $n - 2m = 5$.

3. (25 points)

(a) Prove that for all integers n, n is even if and only if $n^2 + 5$ is odd. Use only the definition of even and odd numbers in your proof.

Proof: Let p : n is even

q : n^2+5 is odd

We will show that $p \to q$ and $q \to p$

Proof of $p \to q$:

Suppose n is even. Then $n = 2k$; $k \in \mathbb{Z}$.

Thus, $n^2 + 5 = (2k)^2 + 5 = 4k^2 + 4 + 1$

$= 2\underbrace{(2k^2+2)}_{\text{an integer b/c } k \in \mathbb{Z}} + 1$

Hence, $n^2 + 5$ is odd.

Proof of $q \to p$:

We prove this by contrapositive, i.e., $\neg p \to \neg q$. Here,

$\neg p$: n is odd

$\neg q$: $n^2 + 5$ is even.

Suppose n is odd. Then $n = 2k+1$; $k \in \mathbb{Z}$

Thus, $n^2 + 5 = (2k+1)^2 + 5 = 4k^2 + 4k + 1 + 5$

$= 4k^2 + 4k + 6 = 2\underbrace{(2k^2 + 2k + 3)}_{\text{an integer b/c } k \in \mathbb{Z}}$

Hence, $n^2 + 5$ is even

(b) Prove that $\sqrt{3}$ is irrational. Explain how your argument would fail if you tried to use it to prove $\sqrt{9}$ irrational.

Proof: For the sake of contradiction, assume $\sqrt{3}$ is rational, i.e.,

$\sqrt{3} = \frac{x}{y}$; x & y are integers and the fraction $\frac{x}{y}$ is in its lowest form.

Then $3y^2 = x^2$, this means that x^2 is a multiple of 3.

// Claim: if x^2 is a multiple of 3, then x is a multiple of 3.

proof of claim: by contrapositive.

Suppose x is NOT a multiple of 3.

thus $x = 3n + r$; where $r = 1, 2$, & $n \in \mathbb{Z}$

$$\Rightarrow x^2 = (3n+r)^2$$
$$= (9n^2 + 6nr + r^2) = 3(3n^2 + 2nr) + r^2.$$
$$= 3m + r^2 \; ; \; m = 3n^2 + 2nr$$

if $r = 1$, $r^2 = 1$ so $x^2 = 3m + 1$;
$r = 2$, $r^2 = 4 = 3 + 1$ so $x^2 = 3(m+1) + 1$ } NOT a multiple of 3.

Hence x^2 is NOT a multiple of 5. //

Thus, x is a multiple of 3. So,

$x = 3n$; $n \in \mathbb{Z}$, therefore

$$3y^2 = x^2 = 9n^2 \Rightarrow y^2 = 3n^2$$

so y^2 is a multiple of 3, and then by previous claim y is a multiple of 3.

This contradicts the fact that $\frac{x}{y}$ was in lowest form, since x, y both have a common factor of 3.

Hence, $\sqrt{3}$ is irrational

If we tried this argument with $\sqrt{9}$,

to get

$$\sqrt{9} = \frac{x}{y}, \text{ so } 9y^2 = x^2$$

The claim

if x^2 is a multiple of 9,

then x is a multiple of 9 is

FALSE, for instance

3^2 is a multiple of 9,

but 3 is NOT a multiple of 9.

Without the Claim, the proof fails.

4. (15 points) Prove or disprove (i.e., give a counterexample to) the following identity for sets A, B, C:

$$(A \cap B) - C = (A - C) \cap (B - C)$$

one can draw the venn diagram to see that the identity is True.

Proof: using the double inclusion of sets.
We will prove that

$$(A \cap B) - C \subseteq (A - C) \cap (B - C)$$

and

$$(A - C) \cap (B - C) \subseteq (A \cap B) - C.$$

Suppose $x \in (A \cap B) - C$. Then $x \in A \cap B$ and $x \notin C$. $x \in A \cap B$ means $x \in A$ and $x \in B$. But we know that $x \notin C$. So $x \in A - C$ and $x \in B - C$. Thus, $x \in (A - C) \cap (B - C)$. Since this holds for every $x \in (A \cap B) - C$, we have

$$(A \cap B) - C \subseteq (A - C) \cap (B - C).$$

Now suppose $x \in (A - C) \cap (B - C)$. Then $x \in A - C$ and $x \in B - C$. It follows that $x \in A$ and $x \in B$ and $x \notin C$. So $x \in A \cap B$ and $x \notin C$. Thus, $x \in (A \cap B) - C$. Since this holds for every $x \in (A - C) \cap (B - C)$, we have

$$(A - C) \cap (B - C) \subseteq (A \cap B) - C.$$

5. (25 points)

(a) **(5pts)** Determine whether each of the following statements is true or false.

(i) $\emptyset \in \emptyset$ False

(ii) $\emptyset \subseteq \emptyset$ True

(iii) $\emptyset \subset \emptyset$ False

(iv) $\emptyset \in \{\emptyset\}$ True

(v) $\emptyset \subseteq \{\emptyset\}$ True

(b) **(5pts)** Determine the cardinality of each of the following sets.

(i) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ 3

(ii) $\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5\}$ 5

(iii) $\{1, \emptyset, a, 2, z, \{1, 2\}, \mathbb{Z}, \{a, z\}, \{a, 1, 2, z\}\}$ 9

(iv) $\{x \mid x \text{ is an odd integer and } 1 \leq x < 22\}$ 11

$= \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21\}$

(v) $\mathcal{P}(\{1, 2, 3\})$

$2^3 = 8.$

(c) **(5pts)** Fill in the blank below using the **logical definition**.

The set A is a subset of set B if an only if

$(\forall x \in U)(x \in A \rightarrow x \in B)$

(d) **(10pts)** Let $A = \{0, 1, 3, 5\}$, $B = \{0, 2, 3, 4, 6\}$, and the universal set is given by $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Find

(i) $A \cup B$

$= \{0, 1, 2, 3, 4, 5, 6\}$

(ii) $A \cap B$

$= \{0, 3\}$

(iii) $A - B$

$= \{1, 5\}$

(iv) $B - A$

$= \{2, 4, 6\}$

(v) $\overline{(\overline{A} \cup B)}$

$= \{1, 5\}$ since $\overline{A} = \{2, 4, 6, 7\}$, and

$\overline{A} \cup B = \{0, 2, 3, 4, 6, 7\}$