Math 150: Discrete Mathematics

Midterm Exam 1- Practice Exam C

NAME (please print legibly): Solutions

Your University ID Number: ______________________

Your University email ______________________

Indicate your instructor with a check in the appropriate box:

Dannenberg	MW 10:25-11:40am	
Kumar	TR 9:40-10:55am	

- You are responsible for checking that this exam has all 8 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers, unless specified otherwise.

Please **COPY** the HONOR PLEDGE and **SIGN:**

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

HONOR PLEDGE:

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YOUR SIGNATURE:______________________

1. (20 points) Prove or disprove (i.e., give a counterexample to) the following identity for sets A, B:

$$(A \cup B) - A = B - (A \cap B).$$

Solution: Proof using the double inclusion of sets. We will prove that

$$(A \cup B) - A \subseteq B - (A \cap B),$$

and

$$B - (A \cap B) \subseteq (A \cup B) - A.$$

Suppose $x \in (A \cup B) - A$. Then $x \in A \cup B$ and $x \notin A$. $x \in A \cup B$ means $x \in A$ or $x \in B$. But we know that $x \notin A$ so we must have $x \in B$. On the other hand, $x \notin A$ implies that $x \notin A \cap B$. Since $x \in B$ and $x \notin A \cap B$, we have $x \in B - (A \cap B)$. Since this holds for every $x \in (A \cup B) - A$, we have $(A \cup B) - A \subseteq B - (A \cap B)$.

Now suppose $x \in B - (A \cap B)$. Then $x \in B$ and $x \notin A \cap B$. Thus, it follows that $x \notin A$. $x \in B$ implies that $x \in A \cup B$. Combining this with $x \notin A$ yields $x \in (A \cup B) - A$. Since this holds for every $x \in B - (A \cap B)$, we have $B - (A \cap B) \subseteq (A \cup B) - A$.

Alternatively:

proof using logical equivalences

$$
\begin{aligned}
x \in (A \cup B) - A &\equiv (x \in A \lor x \in B) \land x \notin A && \text{def}^n \text{ of } \cup, - \\
&\equiv (x \in A \lor x \in B) \land \neg(x \in A) && \text{def}^n \text{ of } \notin \\
\text{Distributive property} \quad &\equiv (x \in A \land \neg(x \in A)) \lor (x \in B \land \neg(x \in A)) \\
&\equiv F \lor (x \in B \land \neg(x \in A)) && \text{Negation law} \\
&\equiv (x \in B \land \neg(x \in A)) && \text{identity law} \\
&\equiv x \in B \land x \notin A \\
&\equiv x \in B \land x \notin A \cap B && \text{otherwise } x \text{ would be in } A. \\
&\equiv x \in B - (A \cap B).
\end{aligned}
$$

2. (10 points) Prove that for all integers n, n is odd if and only if $n^3 + 7$ is even.

Solution: Let

p : n is odd

q : n^3+7 is even

We will show that $p \to q$ and $q \to p$

Proof of $p \to q$:

Suppose n is odd. Then $n = 2k+1$; $k \in \mathbb{Z}$.

Thus, $n^3+7 = (2k+1)^3+7 = 8k^3+12k^2+6k+1+7$

$$= 8k^3+12k^2+6k+8$$

$$= 2(\underbrace{4k^3+6k^2+3k+4}_{\text{an integer b/c } k\in\mathbb{Z}})$$

Hence, n^3+7 is even.

Proof of $q \to p$:

We prove this by contrapositive, i.e.,

$\neg p \to \neg q$. Here,

$\neg p$: n is even

$\neg q$: n^3+7 is odd.

Suppose n is even. Then $n = 2k$; $k \in \mathbb{Z}$.

Thus, $n^3+7 = (2k)^3+7 = 8k^3+7$

$$= 8k^3+6+1 = 2(\underbrace{4k^3+3}_{\text{an integer b/c } k\in\mathbb{Z}})+1$$

Hence, n^3+7 is odd.

3. (20 points) The universe of discourse for all variables below is the set of integers, $\mathbb{Z}$. Determine the truth value of each of the following propositions. **For this problem, you do not need to justify your answers.**

(a) $(\exists n)\,(n^2 < 0)$

FALSE ; $n^2 \geq 0$ always for $n \in \mathbb{Z}$

(b) $(\forall n)\,(n^2 > 0)$

FALSE ; Take $n = 0$.

(c) $(\exists m)(\forall n)\,(n^m = n)$

TRUE . Take $m = 1$.

(d) $(\forall m)(\exists n)\,(n^2 < m)$

FALSE . Take $m = 0$.

(e) $(\forall n)(\exists m)\,(n^2 < m)$

TRUE . Let $n \in \mathbb{Z}$. Take $m = n^2 + 1$.

(f) $(\exists m)(\exists n)\,[(nm = 4) \rightarrow (n + m = -5)]$

TRUE Take $n = -1$, $m = -4$

(g) $(\exists m)(\exists n)\,[(n + m \neq 0) \rightarrow (nm = 1)]$

TRUE. For $m = 3$, $n = -3$, $m + n \neq 0$ is F so conditional is T, regardless of truth value of $mn = 1$.

4. (20 points) Let p, q, r be propositions.

(a) Show that

$$[(\neg p \vee q) \wedge \neg(q \wedge \neg r)] \quad \longrightarrow \quad r \vee \neg p$$

is a tautology. **If you are using a truth table, then you must explain what about your table allows you to conclude the desired result.**

solution: using truth table

let $A = \neg p \vee q$, $B = q \wedge \neg r$, $C = r \vee \neg p$

p	q	r	$\neg p$	A	$\neg r$	B	$\neg B$	$A \wedge \neg B$	C	$(A \wedge \neg B) \to C$
T	T	T	F	T	F	F	T	T	T	T
T	T	F	F	T	T	T	F	F	F	T
T	F	T	F	F	F	F	T	F	T	T
T	F	F	F	F	T	F	T	F	F	T
F	T	T	T	T	F	F	T	T	T	T
F	T	F	T	T	T	T	F	F	T	T
F	F	T	T	T	F	F	T	T	T	T
F	F	F	T	T	T	F	T	T	T	T

since the last column is all True, thus, the given conditional is a tautology.

Alternatively:

using existing logical equivalences:

$\neg p \vee q \equiv p \to q$, cond-disj

$\neg(q \wedge \neg r) \equiv \neg q \vee \neg(\neg r) \equiv \neg q \vee r \equiv q \to r$

De Morgan's — Double Negation

$$[(\neg p \vee q) \wedge \neg(q \wedge \neg r)] \equiv (p \to q) \wedge (q \wedge r)$$

$$\equiv p \to r$$

On the other hand,

$$r \vee \neg p \underset{\text{commutativity}}{\equiv} \neg p \vee r \underset{\text{cond-disj}}{\equiv} p \to r$$

Thus, the given conditional is always True, i.e., is a tautology.

(b) Show that

$$\neg(q \vee (\neg p)) \vee (q \wedge p) \equiv p.$$

If you are using a truth table, then you must explain what about your table allows you to conclude the desired result.

solution: using truth table

Let $q \vee (\neg p) = A$, $q \wedge p = B$

p	q	$\neg p$	A	$\neg A$	B	$\neg A \vee B$
T	T	F	T	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	F	F
F	F	T	T	F	F	F

1st and last column have same truth values, thus, logically equivalent.

Alternatively:

using existing logical equivalences:

$\neg(q \vee (\neg p)) \vee (q \wedge p) \equiv [(\neg q) \wedge (\neg(\neg p))] \vee (q \wedge p)$ by De Morgan's

by Double negation $\equiv ((\neg q) \wedge p) \vee (q \wedge p)$

by commutative law $\equiv (p \wedge \neg q) \vee (p \wedge q)$

by Distributive law $\equiv p \wedge (\neg q \vee q)$

by Negation law $\equiv p \wedge T$

by identity law $\equiv p$

5. (10 points) Prove that $\sqrt{10}$ is irrational.

Solution: For the sake of contradiction, assume that $\sqrt{10}$ is not irrational. Then $\sqrt{10}$ is rational, so

$$\sqrt{10} = \frac{p}{q},$$

where p, q are integers and $q \neq 0$. We also assume that the fraction $\frac{p}{q}$ is in lowest terms, i.e., all common factors have been cancelled. Then

$$\sqrt{10}\, q = p,$$

squaring both sides, we get

$$10\, q^2 = p^2.$$

Thus, p^2 is even. We now prove the following claim:

Claim: "For all integers p, if p^2 is even, then p is even."
Proof of Claim: We establish this result by contrapositive proof. Suppose p is not even, then p is odd. So $p = 2k + 1$ for some integer k. Therefore,

$$p^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since $2k^2 + 2k$ is an integer (because k is an integer), we conclude that p^2 is odd and hence, p^2 is not even. Thus, we have prove that "if p is odd (i.e., not even), then p^2 is odd (i.e., not even)", which is equivalent to "if p^2 is even, then p is even".
Since p is even, we have $p = 2k$ for some integer k. Plugging this value of p into $10\, q^2 = p^2$ yields

$$10\, q^2 = (2k)^2 = 4k^2,$$

hence, $5\, q^2 = 2k^2$. This means that $5\, q^2$ is even, i.e., $5\, q^2$ is a multiple of 2, which implies that q^2 is a multiple of 2, so q^2 is even (since 5 is odd, thus not a multiple of 2). And by previous claim, we see that q is even.
Therefore, we have that p and q are even, i.e., there is a common multiple of 2, which contradicts the fact that the fraction $\frac{p}{q}$ is in the lowest terms. Thus, our initial assumption $\sqrt{10}$ is not irrational is wrong. Hence, we conclude that $\sqrt{10}$ is irrational.

6. (20 points)

(a) **(5pts)** State the definition of the *power set*, $\mathcal{P}(A)$, of a set A.

Solution: The *power set* of a set A is the set whose elements are all the subsets of A.

(b) **(5pts)** Consider the sets: $P = \{1, 4, 9, 16\}$, $Q = \{-2, -1, 0, 1, 2\}$, $R = \{1, 1, 2, 2, 2, 4\}$.

- Compute $P - R$.

  $P - R = \{9, 16\}$

- Compute $Q \cup R$.

  $Q \cup R = \{-2, -1, 0, 1, 2, 4\}$

- Compute $(P \cup R) \cap Q$.

  $(P \cup R) \cap Q = \{1, 2, 4, 9, 16\} \cap Q = \{1, 2\}$

- Compute $|R|$.

  $|R| = 3$

- Compute the power set $\mathcal{P}(R)$.

  $\mathcal{P}(R) = \{\emptyset, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}\}$

(c) **(10pts)** Let A and B be sets inside a universe $\mathcal{U}$ with $|\mathcal{U}| = 30$, $|A| = 12$, $|A \cap B| = 10$ and $|\overline{A \cup B}| = 12$. Find $|B|$.

Solution: We will use the following identity (which is called the Inclusion-Exclusion Principle)

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Observe that

$$|\overline{A \cup B}| = |\mathcal{U}| - |A \cup B|,$$

thus,

$$|A \cup B| = |\mathcal{U}| - |\overline{A \cup B}| = 30 - 12 = 18.$$

Hence,

$$|B| = |A \cup B| - |A| + |A \cap B| = 18 - 12 + 10 = 16.$$