MATH 150 - Fall 2023
Midterm Exam #1 - Practice B - SOLUTIONS

Instructions: Closed book, no notes. Give reasons for all of your answers (except for the individual entries in the truth table in #1), and give a full proof for #4.

1. Use a truth table to show that the following is a tautology:

$$\neg (p \wedge \neg q \wedge (p \to q)) .$$

Soln. Using the definitions, of negation, conditional and triple conjunction to fill the 3rd, 4th and 5th columns, and the negation again to fill the 6th, we obtain

p	q	$\neg q$	$p \to q$	$p \wedge \neg q \wedge (p \to q)$	$\neg(p \wedge \neg q \wedge (p \to q))$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	T	F	T
F	F	T	T	F	T

Since the last column is all "T"s, the compound proposition (statement) is a tautology.

(continued)

2. Determine the truth value of each of the following propositions, if the universe (domain of discourse) is the set $\mathbb{R}$ of all real numbers.

Soln.

(a) $(\exists y \forall x)\,(x^2 = y^3)$

This is False, since there is no one y which is a cube root of every number of the form x^2 (which can be any nonnegative real number).

(b) $(\forall x \exists y)\,(x^2 = y^3)$

This is True, since for every $x \in \mathbb{R}$, $x^2\mathbb{R}$ and every real number has a unique cube root (as can be shown in calculus using the Intermediate Value Theorem (for existence) and the strict monotonicity of $y \to y^3$ (for uniqueness)).

(c) $(\forall x \exists ! y)\,(x^2 = y^3)$

This is also True, by the uniqueness that was described in (b).

(d) $(\forall y \exists x \exists z)\,(x + y = z)$

This is True. For any y, there are many pairs of numbers x, z such that $x + y = z$: let x be any number, and then just define z to be $x + y$.

(e) $(\exists x \forall y \exists z)\,(x + y = z)$

This is True. In fact, any choice of x makes the remaining double-quantified statement true: given any y, just take $z = x + y$.

(f) $(\exists x \exists z \forall y)\,(x + y = z)$

This is False. No matter what x and z are, picking $y \neq z - x$ makes the equation false, so the $\forall y$ is false.

(continued)

3. In the universe (domain of discourse) $U = \mathbb{Z} =$ the integers, let $A = \{0\}$, $B = \{-1, +1\}$, $E =$ the even integers and $O =$ the odd integers. Find each of the following sets:

Soln.

(a) $\overline{E} = \{n \in \mathbb{Z} | n \notin E\} =$ the odd integers $= O$

(b) $E - A = \{n | n \in E \wedge n \notin A\} = \{n | n$ is a nonzero even integer$\}$

(c) $O - A = O$, since $A \cap O = \emptyset$.

(d) $\overline{(A \cup B)} = \{n | n \notin A \cup B\} = \{n | n \neq -1, 0, 1\} = \{n$ s.t. $|n| \geq 2\}$.

4. Prove the following:

Thm. Suppose that $a, b, c, m \in \mathbb{Z}$ are integers such that $a|b$ and $a|m$ but $a \not| c$. Prove that the equation $mx + b = c$ has no solution x in the integers.

Soln. Suppose not. (I.e., proof by contradiction.) Then there does exist an $x \in \mathbb{Z}$ such that $mx + b = c$.

Since $a|b$ and $a|m$, there exist $k, l \in \mathbb{Z}$ such that $b = ka$ and $m = la$.

This implies that

$$c = mx + b = (la)x + (ka) = (lx + k)a.$$

Since $lx + k \in \mathbb{Z}$, this implies that $a|c$, contradicting the assumption that $a \not| c$. Q.E.D.