MATH 150 - FALL 2023

Midterm 1

Practice Exam A - SOLUTIONS

Last Name (Print Family Name): ______________________________
First Name: ______________________________
Student ID#: ______________________________

Circle your instructor and class time:

Matthew Dannenberg Arda Demirhan Allan Greenleaf Anudeep Kumar

Before starting, please read the following instructions completely:

- There are **100 points** in all. You have **75 minutes** to complete this exam.
- Closed book and no notes. The use of notes, calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden. Any phones or electronic devices need to be turned **off** for the duration of the exam.
- Give reasons for all answers. You may use results from class or the book, if clearly cited.
- Sign the following academic honesty statement: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.*

Signature: ______________________________

1. (20 points) Using truth tables, determine if each of the following propositions is a tautology, a contradiction or neither.

(i) (10 points) $(p \to q) \leftrightarrow (\neg p \vee q)$

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since the last column is all "T"s, the compound statement is a tautology.

(ii) (10 points) $\neg(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q)$.

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$	$\neg(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q)$
T	T	T	F	F	F	T	F
T	F	F	T	F	T	F	F
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	F

Since the last column is all "F"s, the compound statement is a contradiction.

2. (20 points) Determine the truth values of each of the following propositions, if the domain of discourse is the set of integers. (5 pts each)

(a) $\forall n(100 \cdot n \geq n^2)$.

The predicate is T for $n = 0$. For $n > 0$, dividing by n preserves the inequality, and it holds iff $100 \geq n$. Thus, it fails for $n \geq 101$, and so the universally quantified statement is F.

(b) $\forall n \forall m \exists \ell(n^2 + m^2 = \ell^2)$.

The statement is F. To show this, it suffices to show $(\exists n \exists m \forall \ell)(n^2 + m^2 \neq \ell^2)$. As one example, if we take, $n = 2$, $m = 2$, then $n^2 + m^2 = 8$, and $\pm\sqrt{8} \notin \mathbb{Z}$.

(c) $\exists n \forall k(nk = n^2)$.

This is T. To show existence, we can take $n = 0$, so that $(\forall k) 0 \cdot k = 0^2$.

(d) $\exists k \forall n(n(n+1) = 2k)$.

This is F. To show this, we need to show $(\forall k \exists n)\big(n(n+1) \neq 2k\big)$. For k fixed, so is $2k$, and we can certainly find an n such that $n(n+1) \neq 2k$. For example, we can take $n = 2k$, so that $n(n+1) = 2k + 4k^2 \neq 2k$ if $k \neq 0$, and $n = 1$ if $k = 0$.

3. (30 points) Consider the following sets:

$$A = \{0, 2, 4, 6, 8\}, B = \{3, 5, 7\} \text{ and } C = \{5, 6, 7, 8, 9\}$$

If $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is the universal set, find the following sets [added: or cardinality in (a)] (10 pts each):

(a) $|A \cup B \cup C|$

$|A \cup B \cup C| = |\{0, 2, 3, 4, 5, 6, 7, 8, 9\}|$, so its cardinality is 9.

(b) $\overline{(\overline{A} \cap \overline{B})}$

As a general set-theoretic identity, by DeMorgan and double-negation we have

$$\overline{(\overline{A} \cap \overline{B})} = \overline{\overline{A \cup B}} = A \cup B.$$

Applying this to these particular A and B, we have $\overline{(\overline{A} \cap \overline{B})} = \{0, 2, 3, 4, 5, 6, 7, 8\}$.

(c) $|(B \times C) - (C \times B)|$. By definition of set difference,

$$(B \times C) - (C \times B) = \{(n, m) | (n \in B \wedge m \in C) \wedge \neg(n \in C \wedge m \in B)\}$$

$$= \{(n, m) | (n \in B \wedge m \in C) \wedge (n \notin C \vee m \notin B)\} \text{ by DeMorgan}$$

$$= \{(n, m) | n \in B \wedge m \in C \wedge n \notin C\} \cup \{(n, m) | n \in B \wedge m \in C \wedge m \notin B\} \text{ by distributivity}$$

$$= \big((B - C) \times C\big) \cup \big(B \times (C - B)\big)$$

$$= \big(\{3\} \times \{5, 6, 7, 8, 9\}\big) \cup \big(\{3, 5, 7\} \times \{6, 8, 9\}\big).$$

Since 3 elements in the first set in the last line occur in the second set, namely $(3, 6), (3, 8), (3, 9)$, the union of the two sets has 5+9-3=11 elements.

4. (30 points)

(a) (10 points) Prove that if A and B are sets, then $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

Proof. $A \subseteq B$ iff $(\forall x)(x \in A \to x \in B)$ iff $(\forall x)(x \notin B \to x \notin A)$ [by Conditional-Contrapositive rule of inference] iff $\overline{B} \subseteq \overline{A}$. Q.E.D.

(b) (10 points) Prove that for all positive numbers a and b, we have

$$a + b \geq 2\sqrt{ab}$$

Proof. Using the fact that $f(x) = x^2$ is increasing on $[0, \infty)$, and thus preserves inequalities, we have, for $a, b > 0$,

$a+b \geq 2\sqrt{ab}$ iff $(a+b)^2 \geq 4ab$ iff $a^2+2ab+b^2 \geq 4ab$ iff $a^2-2ab+b^2 \geq 0$ iff $(a-b)^2 \geq 0$, which is T since $x^2 \geq 0$ for all $x \in \mathbb{R}$. Q.E.D.

(c) (10 points) Prove that for every integer n, n is odd iff $n^2 + 4n + 1$ is even.

Proof. Only if: If n is odd, there exists an integer k such that $n = 2k + 1$, and then $n^2 = 4k^2 + 4k + 1$. Hence,

$$n^2 + 4n + 1 = (4k^2 + 4k + 1) + (8k + 4) + 1 = 4k^2 + 12k + 6$$

$$= 2(2k^2 + 6k + 3),$$

which is even since $2k^2 + 6k + 2 \in \mathbb{Z}$.

If: We proof this by contraposition. To prove that if $n^2 + 4n + 1$ is even then n is odd, it suffices to show that if n is not odd then $n^2 + 4n + 1$ is not even, i.e., if n is even, then $n^2 + 4n + 1$ is odd. However, if n is even, $n = 2l$ for some $l \in \mathbb{Z}$, and then

$$n^2 + 4n + 1 = 4l^2 + 8l + 1 = 2(2l^2 + 4l) + 1$$

is odd. Q.E.D.