Math 150: Discrete Mathematics

Practice Final Exam 2 - Solutions

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Indicate your instructor with a check in the appropriate box:

Dannenberg	MW 10:25-11:40am	
Kumar	TR 9:40-10:55am	

- You are responsible for checking that this exam has all 7 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers, unless specified otherwise.

Please **COPY** the HONOR PLEDGE and **SIGN:**

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

HONOR PLEDGE:

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YOUR SIGNATURE:______________________________

Part A

1. (10 points) Using a truth table, show that $\neg p \vee q \vee (p \wedge \neg q)$ is a tautology.

Solution:

p	q	$\neg p$	$\neg q$	$p \wedge (\neg q)$	$(\neg p) \vee q$	$\neg p \vee q \vee (p \wedge \neg q)$
T	T	F	F	F	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Since the column of the truth table corresponding to our expression is all true, the given expression is a tautology.

2. (15 points) Supoose A and B are sets, $A \oplus B$ is defined to be the set of all x that are in A or B, but not both. Prove that $A \oplus B = A$ if and only if $B = \emptyset$.

Solution: First suppose that $A \oplus B = A$. Suppose for the sake of contradiction that $B \neq \emptyset$. Since $B \neq \emptyset$ we may choose an element $x \in B$. There are two cases: either $x \in A$ or $x \notin A$. If $x \in A$, then $x \in A \cap B$, so $x \notin A \oplus B$ by definition. If $x \notin A$, then since we also have that $x \in B$, we have that $x \in A \oplus B$. But this contradicts the fact that $A \oplus B = A$. Thus, B must be empty.

Now suppose that B is empty. Then $A \cap B = \emptyset$ and $A \cup B = A$. Thus,

$$A \oplus B = (A \cup B) - (A \cap B) = A - \emptyset = A.$$

3. (10 points) Prove that if $x = a \cdot b \cdot c$, where a, b and c are positive real numbers, then either $a \leq x^{1/3}$, $b \leq x^{1/3}$ or $c \leq x^{1/3}$.

Solution: Suppose the conclusion false. Then $a > x^{1/3}$, $b > x^{1/3}$ and $c > x^{1/3}$. Hence, $x = a \cdot b \cdot c > x^{1/3} \cdot x^{1/3} \cdot x^{1/3} = x$, i.e., $x > x$, which is a contradiction.

4. (20 points) Prove each of the following statements.

(i) $x^3 + 2x^2 - 6$ is big-$\mathcal{O}$ of x^3.

Solution: For $x > 2$, we have

$$x^3 + 2x^2 - 6 < x^3 + x \cdot x^2 + x^3 = 3x^3.$$

Hence, with the witnesses $k = 2$ and $C = 3$, we have that $x^3 + 2x^2 - 6$ is big-$\mathcal{O}$ of x^3.

(ii) x^3 is NOT big-$\mathcal{O}$ of x^2.

Solution: If x^3 is big-$\mathcal{O}$ of x^2, then there are positive integers k and C such that for all $x > k$, we have that $x^3 < Cx^2$. Dividing by x^2, this gives $x < C$. But if x is any integer greater than both k and C, it is not true that $x < C$. This is a contradiction. Hence, it is false that x^3 is big-$\mathcal{O}$ of x^3.

5. (15 points) Suppose $p \neq q$ and $r \neq s$ are prime numbers such that $pq^2 = rs^2$.
Choose all the correct options that apply (there could be one or more) and give a brief justification for the option(s) you selected:

(i) $p < r$ might be true since $q > s$ could be true.

False

(ii) $q = r$ is possibly true.

False

(iii) $q = r$ and $p = \frac{s^2}{r}$ must be true.

False

(iv) $q = s$ must be true.

True. Since p, q are prime, pq^2 has a prime factorization $p \cdot q \cdot q$. Similarly, $rs^2 = r \cdot s \cdot s$. By the Fundamental theorem of arithmetic, any number has a unique prime factorization, so since $pq^2 = rs^2$, we must have $p = r$ and $q = s$.

(v) $q = s$ might be true but it is false when $p = 2$.

False

6. (14 points)

(i) What is the binary expansion of $(300)_{10}$?

Solution: We repeatedly divide by 2 to get

$$\begin{aligned} 300 &= 2 \cdot 150 + 0 \\ 150 &= 2 \cdot 75 + 0 \\ 75 &= 2 \cdot 37 + 1 \\ 37 &= 2 \cdot 18 + 1 \\ 18 &= 2 \cdot 9 + 0 \\ 9 &= 2 \cdot 4 + 1 \\ 4 &= 2 \cdot 2 + 0 \\ 2 &= 2 \cdot 1 + 0 \\ 1 &= 2 \cdot 0 + 1 \end{aligned}$$

Thus, $300 = (100101100)_2$.

(ii) Find 9^{300} **mod** 10.

Solution: Observe that $9 \equiv -1 \pmod{10}$. Thus, $9^2 \equiv 1 \pmod{10}$. Hence,

$$9^{300} = 9^{2 \cdot 150} = (9^2)^{150} \equiv 1^{150} \equiv 1 \pmod{10}.$$

7. (16 points) Determine the truth value of each of the following statements, giving reasons. The universe of discourse is the set of integers, $\mathbb{Z}$.

(i) $(\forall\, n)(\exists\, m)\, m = n^3$

True. For any $n \in \mathbb{Z}$, let $m = n^3$.

(ii) $(\exists\, n)(\forall\, m)\, m = n^3$

False. For any $n \in \mathbb{Z}$, there is some $m \in \mathbb{Z}$ such that $m \neq n^3$.

(iii) $(\exists\, k)(\forall\, m)(\exists\, n)\, m = n + k^3$

True. Let $k = 0$. For any $m \in \mathbb{N}$, let $n = m$.

(iv) $(\forall\, m)(\exists\, n)(\exists\, k)\ ((m = n + k^3) \longrightarrow (m = n^3))$.

True. For any $m \in \mathbb{Z}$, choose $n = m$ and $k = 1$. Then $m \neq n + k^3$, so the implication is True.

Part B

8. (10 points) Find all odd integers which have the property that they both leave a remainder of 4 when divided by 7 and leave a remainder of 5 when divided by 11.

Solution: We seek all integers x such that

$$x \equiv 1 \pmod{2}, \quad x \equiv 4 \pmod{7}, \quad x \equiv 5 \pmod{11}.$$

Let $m_1 = 2$, $m_2 = 7$, and $m_3 = 11$, then $m = m_1m_2m_3 = 154$, $M_1 = m_2m_3 = 77$, $M_2 = m_1m_3 = 22$, and $M_3 = m_1m_2 = 14$. Let y_k be an inverse of M_k mod m_k for $k = 1, 2, 3$. So y_1 is an inverse of 77 mod 2, which is equivalent to 1 mod 2, i.e.,

$$77y_1 \equiv 1 \pmod{2} \iff y_1 \equiv 1 \pmod{2}.$$

Thus, $y_1 = 1$ works.
Next we need to find y_2, an inverse of 22 mod 7, which is equivalent to 1 mod 7, i.e.,

$$22y_2 \equiv 1 \pmod{7} \iff y_2 \equiv 1 \pmod{7}.$$

Thus, $y_2 = 1$ works in this case too.
Finally, we find y_3, an inverse of 14 mod 11, which is equivalent to 3 mod 11, i.e.,

$$14y_3 \equiv 1 \pmod{11} \iff 3y_3 \equiv 1 \pmod{11}.$$

Thus, we take $y_3 = 4$.
Therefore, by the Chinese Remainder Theorem, the unique solution mod 154 is

$$x = 1 \cdot M_1y_1 + 4 \cdot M_2y_2 + 5 \cdot M_3y_3 = 1 \cdot 77 \cdot 1 + 4 \cdot 22 \cdot 1 + 5 \cdot 14 \cdot 4 = 77 + 88 + 280 = 445.$$

Since $m = 154$, the unique solution in Z_{154} is: $445 - (2 \cdot 154) = 137$. Hence, the set of all integers x obeying the conditions are all integers that are congruent to 137 mod 154.

9. (10 points) Find the number s such that $0 \leq s \leq 26$ and $13s \equiv 1 \pmod{27}$.

Solution: Observe that $27 = 2 \cdot 13 + 1$ so $1 = 27 - 2 \cdot 13$. Now we use the fact that if $\gcd(a, m) = 1$, then we can find s, t (Bezout coefficients) such that $1 = as + mt$. Then, we have $as \equiv 1 \pmod{m}$. Thus, with $a = 13$ and $m = 27$, we obtain $s = -2$; however, it is not in our range. Hence,

$$-2 \equiv -2 + 27 \pmod{27} = 25 \pmod{27}.$$

Therefore, $s = 25$.

10. (5 points) Associate the letters A, B, ..., Z with the numbers $0, 1, \ldots, 25$. Consider the linear cypher $f : \mathbb{Z}_{26} \to \mathbb{Z}_{26}$ such that $f(p) = p - 7$.

Encrypt the message HELLO WORLD.

Solution:

$$\begin{aligned}
&H \longrightarrow 7 \longrightarrow 0 \longrightarrow A \\
&E \longrightarrow 4 \longrightarrow -3 \longrightarrow\longrightarrow 23 \longrightarrow X \\
&L \longrightarrow 11 \longrightarrow 4 \longrightarrow E \\
&L \longrightarrow 11 \longrightarrow 4 \longrightarrow E \\
&O \longrightarrow 14 \longrightarrow 7 \longrightarrow H \\
&W \longrightarrow 22 \longrightarrow 15 \longrightarrow P \\
&O \longrightarrow 14 \longrightarrow 7 \longrightarrow H \\
&R \longrightarrow 17 \longrightarrow 10 \longrightarrow K \\
&L \longrightarrow 11 \longrightarrow 4 \longrightarrow E \\
&D \longrightarrow 3 \longrightarrow -4 \longrightarrow\longrightarrow 22 \longrightarrow W
\end{aligned}$$

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11. (10 points) Give a proof by induction that $n! > \frac{1}{4} \cdot 3^n$ for all integers $n \geq 4$.

Solution: <u>*Base step:*</u> If $n = 4$, then $4! = 24$ and $\frac{1}{4} \cdot 3^4 = \frac{81}{4} = 20 + \frac{1}{4} < 24 = 4!$. Thus, the given inequality is true.
<u>*Induction step:*</u> Assume for some $k \in \mathbb{N}$ such that $k \geq 4$, we have $k! > \frac{1}{4} \cdot 3^k$. We now want to show that

$$(k+1)! > \frac{1}{4} \cdot 3^{k+1}.$$

Observe that

$$(k+1)! = (k+1) \cdot k! > (k+1) \cdot \frac{1}{4} \cdot 3^k,$$

where the inequality follows from the inductive hypothesis. Since $k \geq 4$, we have $k + 1 \geq 5 > 3$, so

$$(k+1)! > 3 \cdot \frac{1}{4} \cdot 3^k = \frac{1}{4} \cdot 3^{k+1},$$

as desired. This proves that $n! > \frac{1}{4} \cdot 3^n$ for all integers $n \geq 4$.

12. (15 points)

(i) If one has 16 basketball teams in a tournament which requires the teams are first put into 4 groups (of 4 teams). How many ways are there to form the first group?

Solution: There are 16 teams and we are choosing 4. Thus, there are

$$C(16, 4) = \frac{16!}{12!\,4!}$$

ways to form the first group.

(ii) Given the first group has been picked, how many ways are there to form the second group?

Solution: As only 12 teams remain, there are

$$C(12, 4) = \frac{12!}{8!\,4!}$$

ways to to form the second group.

(iii) Thus, how many distinct ways are there to set up all four groups?

Solution: The number of distinct ways to set up all four groups are

$$C(16, 4) \cdot C(12, 4) \cdot C(8, 4) \cdot 1 = \frac{16!\,12!\,8!}{12!\,4!\,8!\,4!\,4!\,4!} = \frac{16!}{(4!)^4}.$$