Math 150: Discrete Mathematics

Practice Final Exam 1 - Solutions

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Indicate your instructor with a check in the appropriate box:

Dannenberg	MW 10:25-11:40am	
Kumar	TR 9:40-10:55am	

- You are responsible for checking that this exam has all 8 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers, unless specified otherwise.

Please **COPY** the HONOR PLEDGE and **SIGN:**

I affirm that I will not give or receive any unauthorized help on this exam,
and all work will be my own.

HONOR PLEDGE:

____

____

____

YOUR SIGNATURE:______________________________

Part A

1. (12 points) Find the binary and hexadecimal representation for the number with decimal representation 234.

Solution:

$234 = 2 \cdot 117 + 0$
$117 = 2 \cdot 58 + 1$
$58 = 2 \cdot 29 + 0$
$29 = 2 \cdot 14 + 1$
$14 = 2 \cdot 7 + 0$
$7 = 2 \cdot 3 + 1$
$3 = 2 \cdot 1 + 1$
$1 = 2 \cdot 0 + 1$

Thus, $234 = (11101010)_2$.

Grouping the binary expansion into groups of 4: 1110 1010, converting to hexadecimal digits, we get $(EA)_{16}$.

2. (12 points) Show that $[P \wedge (\neg Q \vee \neg R)] \rightarrow (P \rightarrow \neg Q)$ is neither a contradiction nor a tautology.

Solution: Let $\mathbf{A} : P \wedge (\neg Q \vee \neg R)$ and $\mathbf{B} : P \rightarrow \neg Q$. Thus, we want to prove that $\mathbf{A} \rightarrow \mathbf{B}$ is neither a contradiction nor a tautology.

P	Q	R	¬Q	¬R	$\neg Q \vee \neg R$	**A**	**B**	$\mathbf{A} \rightarrow \mathbf{B}$
T	T	T	F	F	F	F	F	T
T	T	F	F	T	T	T	F	F
T	F	T	T	F	T	T	T	T
T	F	F	T	T	T	T	T	T
F	T	T	F	F	F	F	T	T
F	F	T	T	F	T	F	T	T
F	T	F	F	T	T	F	T	T
F	F	F	T	T	T	F	T	T

Observing the last column we conclude that neither all cases are True and nor False. Hence, it is neither a tautology nor a contradiction.

3. (12 points) Let P, Q, R be statements. Prove that

$$\neg[P \to (Q \wedge R)] \equiv (\neg(P \to Q) \vee (\neg(P \to R)).$$

Solution: We start with conditional-disjunction identity: $P \to Q \equiv (\neg P) \vee Q$. Thus,

$$\begin{aligned}
\neg[P \to (Q \wedge R)] &\equiv \neg[(\neg P) \vee (Q \wedge R)] \\
&\equiv P \wedge (\neg(Q \wedge R)) && \text{(DeMorgan's Law)} \\
&\equiv P \wedge ((\neg Q) \vee (\neg R)) && \text{(DeMorgan's Law)} \\
&\equiv (P \wedge (\neg Q)) \vee (P \wedge (\neg R)) && \text{(Distributive Law)} \\
&\equiv \neg((\neg P) \vee Q)) \vee \neg((\neg P) \vee R)) && \text{(DeMorgan's Law)} \\
&\equiv (\neg(P \to Q) \vee (\neg(P \to R)) && \text{(conditional-disjunction)},
\end{aligned}$$

as desired.

4. (12 points) Prove or disprove the the following identity:

$$A \times (B \setminus C) = (A \setminus B) \times (A \setminus C).$$

Solution: This is False.

Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$, $C = \{1\}$.
Then $B \setminus C = \{2\}$, thus $A \times (B \setminus C) = \{(1, 2), (2, 2), (3, 2)\}$.
And, $A \setminus B = \{3\}$, $A \setminus C = \{2, 3\}$, thus $(A \setminus B) \times (A \setminus C) = \{(3, 2), (3, 3)\}$.
Hence, $A \times (B \setminus C) \neq (A \setminus B) \times (A \setminus C)$.

Let (x, y) be an arbitrary element of $A \times (B \setminus C)$. We can conclude that $x \in A$ and $y \in B \setminus C$, i.e., $y \in B$ and $y \notin C$.

Now consider (x, y) be an arbitrary element of $(A \setminus B) \times (A \setminus C)$. This means that $x \in A \setminus B$ and $y \in A \setminus C$. We can conclude that $x \in A$ and $x \notin B$, which means that $A = A \setminus B$, implying that $B = \emptyset$. On the other hand, $y \in A$ and $y \notin C$, which means that $B \setminus C = A \setminus C$, i.e., $B = A$. This is impossible.

5. (12 points) Determine if $3x^3 \ln(x^2) + x^2 - 3$ is $\Omega(x^3)$.

Solution: We want to show that $\exists C, k > 0$ such that for all $x > k$

$$|3x^3 \ln(x^2) + x^2 - 3| \geq C|x^3|.$$

Let $k = e$, observe that

$$3x^3 \ln(x^2) + x^2 - 3 = 6x^3 \ln x + x^2 - 3 \geq 6x^3 + x^2 - 3$$

for $x \geq e$. And since $x^2 - 3 > 0$ for $x \geq e$, we see that $6x^3 + x^2 - 3 \geq 6x^3$. Hence, $3x^3 \ln(x^2) + x^2 - 3$ is $\Omega(x^3)$ for witnesses $C = 6$ and $k = e$.

6. (14 points) Let n be an integer. Prove that $(n+1)^2 + 6$ is even if and only if n odd.

Solution: ($\Longrightarrow$) We prove the contrapositive, i.e., if n is even, then $(n+1)^2 + 6$ is odd. Let $n = 2k$ for $k \in \mathbb{Z}$. Then

$$(n+1)^2 + 6 = (2k+1)^2 + 6 = 4k^2 + 4k + 1 + 6 = 2(2k^2 + 2k + 3) + 1.$$

Thus, $(n+1)^2 + 6$ is odd since $2k^2 + 2k + 3 \in \mathbb{Z}$ because $k \in \mathbb{Z}$.

($\Longleftarrow$) We prove this with a direct proof. Suppose n is odd, i.e., $n = 2k+1$ for $k \in \mathbb{Z}$. Then

$$(n+1)^2 + 6 = (2k+1+1)^2 + 6 = (2k+2)^2 + 6 = 4(k+1)^2 + 6 = 2(2(k+1)^2 + 3)$$

Thus, $(n+1)^2 + 6$ is even since $2(k+1)^2 + 3 \in \mathbb{Z}$ because $k \in \mathbb{Z}$.

7. (16 points) Determine the truth value of each of the following statements, giving reasons. The universe of discourse is the set of integers, $\mathbb{Z}$.

(i) $(\forall\, n)\, n^2 > 0$
Solution: False. Take $n = 0$.

(ii) $(\exists\, m)(\forall\, n)\, n^m = n$
Solution: True. $m = 1$.

(iii) $(\forall\, m)(\exists\, n)\, n^2 < m$
Solution: False. Take m to be negative.

(iv) $(\exists\, m)(\exists\, n)\ ((n \cdot m = 4) \longrightarrow (n + m = -5))$.
Solution: True. $m = -1$ and $n = -4$.

8. (10 points) Find $7^{530} \pmod{23}$.

Solution: By Fermat's Little Theorem, we have $7^{22} \equiv 1 \pmod{23}$. Therefore,

$$7^{530} = 7^{22\cdot 24+2} = (7^{22})^{24} \cdot 7^2 \equiv 1^{22} \cdot 49 \pmod{23} \equiv 3 \pmod{23}.$$

Thus, $7^{530} \pmod{23} = 3$.

Part B

9. (10 points) Recall that $\mathbb{Z}_n$ denotes the set of integers $\{0, 1, \ldots, n-1\}$. Determine if the function $f : \mathbb{Z}_{175} \to \mathbb{Z}_{175}$ given by $f(x) = (32x + 4) \bmod 175$ is invertible. If so, find its inverse.

Solution: Let $y = f(x) \in \mathbb{Z}_{175}$. Then $y \equiv (32x + 4) \bmod 175$, i.e., $y - 4 \equiv 32x \bmod 175$. The function is invertible if $\gcd(32, 175) = 1$. Applying the Euclid's algorithm:

$$\begin{aligned}
175 &= 32 \cdot 5 + 15 \\
32 &= 15 \cdot 2 + 2 \\
15 &= 2 \cdot 7 + 1 \\
2 &= 1 \cdot 2 + 0.
\end{aligned}$$

Therefore, $\gcd(32, 175) = 1$ and so the function is invertible. Using back substitution:

$$\begin{aligned}
1 &= 15 - (2 \cdot 7) = 15 - [32 - (15 \cdot 2)]7 \\
&= (15 \cdot 15) - (32 \cdot 7) = 15[175 - (32 \cdot 5)] - (32 \cdot 7) \\
&= (175 \cdot 15) + (32 \cdot (-82)).
\end{aligned}$$

The last expression tells us that $32(-82) = 175(-15) + 1$, this implies that

$$32(-82) \equiv 1 \pmod{175}.$$

And we conclude that -82 is an inverse of 32 modulo 175. Thus,

$$-82(y - 4) \equiv x \pmod{175},$$

equivalently

$$x \equiv 93y - 372 \equiv 93y - 22 \pmod{175},$$

where we have used the fact that $-82 \equiv 93 \pmod{175}$ and $372 \equiv 22 \pmod{175}$. Hence,

$$f^{-1}(y) = 93y - 22 \pmod{175}$$

is the inverse of f. Since f has an inverse, it is invertible.

10. (10 points) Find all integers x satisfying

$$x^2 - 2x + 1 \equiv 0 \pmod{49}.$$

(Hint: Be careful! 49 is not prime!)

Solution: The given congruence can be written as

$$(x-1)^2 \equiv 0 \pmod{7^2}.$$

Thus, $(x-1)^2 = 7k$ for some $k \in \mathbb{Z}$, which can be re-written as

$$\left(\frac{x-1}{7}\right)^2 = k.$$

Since $k, x \in \mathbb{Z}$, then $\frac{x-1}{7}$ must be as well, because otherwise this would be a non-integer rational, whose square cannot be an integer. Hence,

$$\frac{x-1}{7} = l$$

for some $l \in \mathbb{Z}$. Thus, $x - 1 = 7l$, which yields that the solutions are given by

$$x = 7l + 1 \text{ for } l \in \mathbb{Z}.$$

11. (10 points) Prove that: For all integers $n > 0$ it holds that

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

Solution: Base step: if $n = 1$, we have $\frac{1}{1\times 2} = \frac{1}{2} = \frac{1}{1+1}$, which is clearly true.

Inductive step: Assume that the given statement is true for some fixed integer k, i.e., it holds that

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{k \times (k+1)} = \frac{k}{k+1}.$$

Need to show that

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{k \times (k+1)} + \frac{1}{(k+1) \times (k+2)} = \frac{k+1}{k+2}.$$

Observe that

$$\begin{aligned}
\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{k \times (k+1)} + \frac{1}{(k+1) \times (k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1) \times (k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}.
\end{aligned}$$

Hence, by induction

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

holds true for all integers $n > 0$.

12. (10 points)

(a) How many bit strings of length 20 either begin with "111" or do not contain the substring "01".

Solution: Begin with "111": 2^{17} possible strings. Do not contain "01": 21 but 18 have been counted under starting with "111" so deducting the overlap, we are left with 3 possible strings (000....0, 100....0, 110....0). Total strings $= 2^{17} + 3$.

(b) Prove: Among ten people, at least two were born on the same day of the week.

Solution: By generalized pigeonhole principle, $\lceil \frac{10}{7} \rceil = 2$

13. (10 points)

(a) What is the coefficient of $x^{32}y^{79}$ in the expansion of $(17x - 20y)^{111}$?

Solution: $C(111, 79) \cdot (17)^{32} \cdot (-20)^{79}$.

(b) A coin is flipped 12 times. How many ways are there of obtaining exactly 4 heads?

Solution: $C(12, 4) = 495$ ways (since coin flips are not distinct)