Math 150: Discrete Mathematics
Midterm Exam 1 Practice

Thursday, May 30, 2024

NAME (please print legibly): ____________________

Your University ID Number: ____________________

Your University Email: ____________________

Indicate your instructor with a check in the appropriate box:

Gotshall	

- You are responsible for checking that this exam has all **blank** pages.
- No calculators, phones, electronic devices, books, or notes are allowed during the exam.
- Show all work and justify all answers, unless specified otherwise.

Please **COPY** the HONOR PLEDGE and **SIGN:**
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

HONOR PLEDGE:

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YOUR SIGNATURE ____________________

1. (15 points) Let p, q, r be propositions. Use a truth table to determine whether $p \to (q \to r)$ is logically equivalent to each of the following:

- $(p \to q) \to r$
- $q \to (\neg p \vee r)$

Be sure to explain what parts of the truth table justifies your claims.

Solution: We write out the truth table, considering all possible values of the independent Boolean variables p, q, and r. For brevity, we include all relevant entries for both parts,

p	q	r	$p \to q$	$(p \to q) \to r$	$q \to r$	$p \to (q \to r)$	$\neg p$	$\neg p \vee r$	$q \to (\neg p \vee r)$
T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	F	F
T	F	T	F	T	T	T	F	T	T
F	T	T	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T	T	T
T	F	F	F	T	T	T	F	F	T
F	F	F	T	F	T	T	T	T	T

Due to the red values in the columns for $(p \to q) \to r$ and $p \to (q \to r)$, we see these two propositions are NOT logically equivalent (since they take different truth values in some cases). As the columns for $p \to (q \to r)$ and $q \to (\neg p \vee r)$ have the same truth values in all cases, we conclude these two propositions are logically equivalent.

2. (20 points) Determine the truth value of each of the following propositions, when the universes of discourse under consideration for each part are as written below **(consider each case separately)**.

(a) $(\exists x)\left(x^2 - \frac{1}{3} = 0\right)$ for **Case 1:** $x \in \mathbb{R}$ and **Case 2:** $x \in \mathbb{Q}$.

Solution: Case 1: Let $x = 1/\sqrt{3}$. Since this is a real number, the statement is true.
Case 2: The statement is false. BWOC say we can find a solution $x = a \in \mathbb{Q}$ to the equation. Then we solve to get $a = \pm 1/\sqrt{3}$. By definition of $\mathbb{Q}$, $\exists\ c, d \in \mathbb{Z}$ and $d \neq 0$ such that $1/\sqrt{3} = c/d$. We consider two cases.
Case 1: $c = 0$. Then $1/\sqrt{3} = 0$, which is a contradiction.
Case 2: $c \neq 0$. Then $\sqrt{3} = d/c \in \mathbb{Q}$ by definition. Recall from class that $\sqrt{n}$ is either an integer or irrational. Since it is not an integer, we have $\sqrt{3}$ is irrational, giving us our contradiction.
Hence there is no rational solution to $x^2 - 1/3 = 0$.

(b) $(\forall x)\left(x > x - \frac{5}{6}\right)$ for **Case 1:** $x \in \mathbb{R}$ and **Case 2:** $x \in \mathbb{Q}$ and **Case 3:** $x \in \mathbb{Z}$.

Solution: Case 1: The statement is clearly true by basic algebra. If a universal quantifier is always true for a given domain, it certainly remains true when we shrink the domain (if the equation holds for every $x \in \mathbb{R}$, it has to hold for every $x \in \mathbb{Q}$ and $x \in \mathbb{Z}$, as $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$). Hence Cases 2 and 3 are also true.

(c) $(\forall x)(\exists y)\left(\frac{x}{y} = \sqrt{x}\right)$ for **Case 1:** $x, y \in \mathbb{R}^+$ and **Case 2:** $x, y \in \mathbb{Z}^+$.

Solution: Case 1: The statement is true. Given any $x \in \mathbb{R}^+$, let $y = \sqrt{x}$ (which is a real number since $x > 0$). Observe that $x/\sqrt{x} = \sqrt{x}$.
Case 2: The statement is false. Recall $\neg(\forall x)(\exists y)P(x, y) \equiv (\exists x)(\forall y)\neg P(x, y)$. So let $x = 2$. Setting $2/y = \sqrt{2}$, we solve for $y = 2/\sqrt{2} = \sqrt{2}$. Clearly $\sqrt{2} \notin \mathbb{Z}^+$, justifying our claim.

(d) $(\exists y)(\forall x)\left(\frac{x}{y} = \sqrt{x}\right)$ for **Case 1:** $x, y \in \mathbb{R}^+$ and **Case 2:** $x, y \in \mathbb{Z}^+$.

Solution. Case 1: The statement is false. To see this, rewrite the equation as $y = \sqrt{x}$ and observe that clearly a fixed y value will not work for every single $x \in \mathbb{R}^+$.
Case 2: The statement is false. One way to see this is to use the same argument as in Case 1, since it is clear that no fixed $y \in \mathbb{Z}^+$ will equal $\sqrt{x}$ for every single $x \in \mathbb{Z}^+$. Alternatively,

recall that $(\exists y)(\forall x)P(x,y) \implies (\forall x)(\exists y)P(x,y)$. Taking the contrapositive, we find that $\neg((\forall x)(\exists y)P(x,y)) \implies \neg((\exists y)(\forall x)P(x,y))$. Since we saw in part (c) that this hypothesis is true when $x, y \in \mathbb{Z}^+$, it follows that part (d) is false (since its negation is true).

Some Proofs to Practice

1. Show that if $x, y \in \mathbb{Z}$, xy is even, and $x + y$ is even, then both x and y are even.
2. Prove that if $a, b \in \mathbb{R}$ and $a \neq 0$, then $\exists! r \in \mathbb{R}$ such that $ar + b = 0$.
3. Show $\nexists\ x, y \in \mathbb{Z}^+$ (there does not exist a positive integral solution) to the equation $x^2 + y^4 = 100$.
4. Show $\forall n \in \mathbb{Z}$ that n is even $\leftrightarrow n^3 + 3$ is odd.
5. Prove that if n is an integer, these four statements are equivalent:
   (a) n is even
   (b) $n + 1$ is odd
   (c) $3n + 1$ is odd
   (d) $3n$ is even

Solution: Proof 1 is Example 7 in Section 1.8 of the book. Proof 2 is Example 13 in Section 1.8 of the book.

Proof of 3. BWOC say $\exists\ x, y \in \mathbb{Z}$ satisfying $x^2 + y^4 = 100$. Since both summands are positive due to the even exponents, we must have $x^2 \leq 100$ and $y^4 \leq 100$. Hence the only possibilities are $|x| \leq 4$ and $|y| \leq 3$. While this technically gives us nine cases to check, we actually only have to check the following four,

Case 1: $x = 4$ and $y = 3$. Then we get $4^3 + 3^4 = 145$.

Case 2: $x = 4$ and $y = 2$. Then we get $4^3 + 2^4 = 80$.

Case 3: $x = 3$ and $y = 3$. Then we get $3^3 + 3^4 = 108$.

Case 4: $x = 2$ and $y = 3$. Then we get $2^3 + 3^4 = 89$.

We are actually done at this point because any x, y pair where $y < 2$ will give a number smaller than in Case 2, which is already too small. Similarly, any x, y pair where $x < 2$ will give a number smaller than in Case 4, which is too small. $\square$

Proof of 4. ($\Longrightarrow$) Assume n is even. By definition, $n = 2k$ for some $k \in \mathbb{Z}$. Notice that

$$n^3 + 3 = (2k)^3 + 3 = 8k^3 + 2 + 1 = 2(4k^3 + 1) + 1.$$

Since $(4k^3 + 1) \in \mathbb{Z}$, by definition $n^3 + 3$ is odd.

($\Longleftarrow$) We prove the contrapositive (note a direct proof would be very difficult). So suppose n is odd. By definition, $n = 2k + 1$ for some $k \in \mathbb{Z}$. We compute

$$n^3 + 3 = (2k + 1)^3 + 3 = 8k^3 + 4k^2 + 2k + 1 + 3 = 2(4k^3 + 2k^2 + k + 2).$$

Since $4k^3 + 2k^2 + k + 2$ is an integer, by definition $n^3 + 3$ is even. This finishes the proof. $\square$

Proof of 5. We need to show the following chain of equivalences,

$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (a).$$

$(a) \Longrightarrow (b)$. Assume n is even. By definition, $n = 2k$ for some $k \in \mathbb{Z}$. Observe that $n + 1 = 2k + 1$. By definition, $n + 1$ is odd.

$(b) \Longrightarrow (c)$. Suppose $n+1$ is odd. By definition, $n+1 = 2k+1$ for some $k \in \mathbb{Z}$. Equivalently, $n = 2k$. So $3n + 1 = 3(2k) + 1 = 2(3k) + 1$. As $3k \in \mathbb{Z}$, by definition $3n + 1$ is odd.

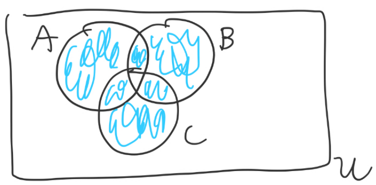
$(c) \Longrightarrow (d)$. Assume $3n + 1$ is odd. By definition, $3n + 1 = 2k + 1$ for some $k \in \mathbb{Z}$. Equivalently, $3n = 2k$. Hence $3n$ is even by definition.

$(d) \Longrightarrow (a)$. We prove the contrapositive (note a direct proof would be very difficult). So assume n is odd. By definition, $n = 2k + 1$ for some $k \in \mathbb{Z}$. We compute $3n = 3(2k + 1) = 6k + 3 = 2(3k + 1) + 1$. Since $3k + 1 \in \mathbb{Z}$, by definition $3n$ is odd. This finishes the proof. $\square$

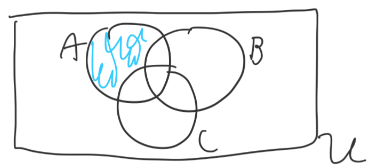
4. Prove or disprove the following set identities,

- $(A \cup B \cup C) - (A \cap B \cap C) = \overline{A \cap B \cap C} - \overline{A \cup B \cup C}$
- $A - (B \cup C) = \emptyset$
- Redo the previous part with the added assumption that $A \subseteq B \cup C$.

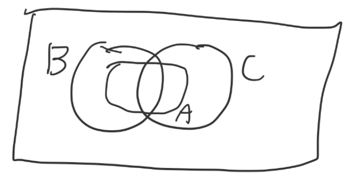
Solution: Recall set identities can be easily checked by drawing out the Venn Diagram. One can check (DO make sure you can make these diagrams on your own)



$(A \cup B \cup C) - (A \cap B \cap C)$
$= \overline{A \cap B \cap C} - \overline{A \cup B \cup C}$



$A - (B \cup C)$



$A - (B \cup C)$
where $A \subseteq B \cup C$

Hence we see the first identity is true, the second is false in general but true in the case $A \subseteq B \cup C$. Now we need to create the precise arguments. Recall there are three main ways to prove set identities. The simplest is the following,

Proof of the first identity. Note that

$$\begin{aligned} x \in (A \cup B \cup C) - (A \cap B \cap C) &\iff (x \in A \cup B \cup C) \wedge (x \notin A \cap B \cap C) \\ &\iff (x \notin \overline{A \cup B \cup C}) \wedge (x \in \overline{A \cap B \cap C}) \\ &\iff x \in \overline{A \cap B \cap C} - \overline{A \cup B \cup C}. \end{aligned}$$

□

To disprove the second identity, we need an explicit counterexample. The middle Venn Diagram tells us any $A \not\subseteq B \cup C$ will do the trick. So let $A = \{1, 2\}$, $B = \{3, 4\}$ and $C = \{5, 6\}$. In this case, $A - (B \cup C) = A \neq \emptyset$ since A and $B \cup C$ are disjoint.

Switching to the third item (or the third Venn Diagram), **note proving a set is empty requires contradiction**, so that we can pick an x in our set and apply the proof techniques we learned.

Proof of the third identity. BWOC say $\exists\ x \in A - (B \cup C)$. By definition, $x \in A$ and $x \notin B \cup C$. Since $A \subseteq B \cup C$, we have $x \in A$ implies $x \in B \cup C$. This gives us our contradiction. $\square$

5. NO JUSTIFICATION NEEDED ON ANY PART

(a) Determine whether each of the following statements is true or false, where A is any set.

- $A \in A$ False
- $A \subseteq A$ True
- $A \subset A$ False
- $A \in \{A\}$ True
- $A \subseteq \{A\}$ False
- $\{A\} \in \{\{1, 2, \{A\}\}\}$ False because the right side contains the single element $\{1, 2, \{A\}\}$
- $\{\{A\}\} \in \{1, 2, \{A\}\}$ False because the right set contains the elements 1, 2, and $\{A\}$. None of these equals $\{\{A\}\}$.
- $\{\{A\}\} \subset \{1, 2, \{A, 4\}\}$ False because the right set contains the elements 1, 2, and $\{4, A\}$. None of these equals $\{\{A\}\}$.

(b) Determine the cardinality of each of the following sets (the answer might be ∞).

- $\{A, \{A\}, \{A, \{A\}\}\}$ where A is any set 3 because the elements are A, $\{A\}$, and $\{A, \{A\}\}$.
- $\{0, \emptyset, \mathbb{C}\}$ 3 because the elements are $0, \emptyset, \mathbb{C}$.
- $\mathcal{P}(\{0, \emptyset\})$ $|\{0, \emptyset\}| = 2$. Hence $|\mathcal{P}(\{0, \emptyset\}| = 2^2 = 4$.

(c) Finish the sentence below by giving the appropriate logical definition (i.e. find a condition on each set element equivalent to the given condition on the sets and then express this symbolically with the appropriate quantifiers and predicates),

The set A is a subset of set B if and only if
Solution: $(\forall x \in A)(x \in B)$. Also correct: $(x \in A) \to (x \in B)$.

(d) Let $A = \{1, 2, 3, \{1, 2, 3\}\}$, $B = \{4, 5, 6, \{1, 2\}, 1\}$, and the universal set be given by $U = \{1, 2, 3, 4, 5, 6, \{1, 2, 3\}, \{1, 2\}, \{1, 2, 3, \{4, 5, 6\}\}\}$. Find

- $A \cup B$
  Solution: $A \cup B = \{1, 2, 3, \{1, 2, 3\}, 4, 5, 6, \{1, 2\}\}$.

- $A \cap B$
  Solution: $A \cap B = \{1\}$, as this is the only element they have in common.

- $A - B$
  Solution: $A - B = \{2, 3, \{1, 2, 3\}\}$ (we removed $1 = A \cap B$).

- $B - A$
  Solution: $B - A = \{4, 5, 6, \{1, 2\}\}$ (we removed $1 = A \cap B$)

- $\overline{A \cap B}$
  Solution: $\overline{A \cap B} = U - (A \cap B) = U - \{1\} = \{2, 3, 4, 5, 6, \{1, 2, 3\}, \{1, 2\}, \{1, 2, 3, \{4, 5, 6\}\}\}$