Math 143

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<u>goal</u>: get power series expansions for as many functions as possible.

<u>Taylor series</u>:

<u>Q</u>. If $f(x)$ has a power series expansion about $x=a$

$$C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + C_4(x-a)^4 + \cdots$$

on $|x-a| < R$, what must the coefficients be in terms of $f(x)$ and a?

<u>A</u>. ① $f(a) = C_0 + \cancel{C_1 \cdot 0} + \cancel{C_2 \cdot 0^2} + \cancel{C_3 \cdot 0^3} + \cdots$

$$\boxed{C_0 = f(a)}$$

② $f'(x) = 0 + C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + 4C_4(x-a)^3 + \cdots$

$$f'(a) = c_1 + \cancel{2c_2 \cdot 0} + \cancel{3c_3 \cdot 0^2} + \cancel{4c_4 \cdot 0^3} + \cdots$$

$$\boxed{c_1 = f'(a)}$$

<u>tangent line appx.</u>

$f(x) \approx f(a) + f'(a)(x-a)$

for x near a

③ $f''(x) = 0 + 2c_2 + 2\cdot3\cdot c_3(x-a) + 3\cdot4\cdot c_4(x-a)^2 + 4\cdot5\cdot c_5(x-a)^3 + \cdots$

$$f''(a) = 2c_2 + \cancel{2\cdot3\cdot c_3\cdot 0} + \cancel{3\cdot4\cdot c_4\cdot 0^2} + \cdots$$

$$\boxed{c_2 = \frac{f''(a)}{2}}$$

④ $f'''(x) = 0 + 2\cdot3\cdot c_3 + 2\cdot3\cdot4\cdot c_4(x-a) + 3\cdot4\cdot5\cdot c_5(x-a)^2 + \cdots$

$$f'''(a) = 3!\, c_3$$

$$\boxed{c_3 = \frac{f^{(3)}(a)}{3!}}$$

$f^{(n)}(a) = n^{th}$ deriv. of f at a

$\vdots$

⑤ $f^{(n)}(a) = 2\cdot3\cdot4\cdot\cdots n\cdot c_n = \boxed{n!\, c_n = f^{(n)}(a)}$

$$\Rightarrow \boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Thm. If $f(x)$ has a power series expansion about $x=a$,

i.e., $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ for $|x-a|<R$

then the coefficients are given by

the formula $c_n=\dfrac{f^{(n)}(a)}{n!}$.

A. So the answer to our original question is:

If $f(x)$ has a power series expansion at $x=a$ then

it must be of the form

$$\boxed{f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n}$$

$$=\frac{f(a)}{0!}+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$+\frac{f'''(a)}{3!}(x-a)^3+\frac{f''''(a)}{4!}(x-a)^4+\dots$$

DEF. This power series expansion of $f(x)$ is called

the Taylor series of $f(x)$ at $x=a$.

DEF. The common case of $a=0$ is called the

Maclaurin series of $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

ex/ Find the Maclaurin series (about $x=0$) for $f(x)=e^x$ and its radius of convergence:

$f(x) = e^x$ $\quad$ $f(0) = 1$ $\quad$ $c_0 = 1/1$ $\leftarrow 0!$

$f'(x) = e^x$ $\quad$ $f'(0) = 1$ $\quad$ $c_1 = 1/1$ $\leftarrow 1!$

$f''(x) = e^x$ $\quad$ $f''(0) = 1$ $\quad$ $c_2 = 1/2$ $\leftarrow 2!$

$f'''(x) = e^x$ $\quad$ $f'''(0) = 1$ $\quad$ $c_3 = 1/6$ $\leftarrow 3!$

$\downarrow$

$c_n = 1/n!$

So the Taylor series for $f(x)=e^x$ at $x=0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \boxed{\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots = e^x}$$

Radius of convergence: (this is ex2 from the first day on power series)

ratio test: $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$

so $L < 1$ for all x-values so $\boxed{R = \infty \text{ and IOC} = (-\infty, \infty)}$

notes: ① Test for Div. $\Rightarrow$ $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x

so $n!$ "grows faster" than x^n for any x.

② We also showed that $e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} 1^n$

so $e = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$

③ And, e.g., $\sum_{n=0}^{\infty} \frac{9^n}{11^n n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{9}{11}\right)^n = e^{9/11}$

$x = \frac{9}{11}$

This series converges, but what is the value of its sum? $e^{9/11}$

ex2/ Find the Taylor series expansion for $f(x) = e^{2x}$ about $x = -4$ its radius of convergence:

$f(x) = e^{2x}$	$f(-4) = 1e^{-8}$	$c_0 = 1e^{-8}/1!$
$f'(x) = 2e^{2x}$	$f'(-4) = 2e^{-8}$	$c_1 = 2e^{-8}/1!$
$f''(x) = 2^2 e^{2x}$	$f''(-4) = 2^2 e^{-8}$	$c_2 = 2^2 e^{-8}/2!$
$f'''(x) = 2^3 e^{2x}$	$f'''(-4) = 2^3 e^{-8}$	$c_3 = 2^3 e^{-8}/3!$

$c_n = 2^n e^{-8}/n!$

So the Taylor is $\sum_{n=0}^{\infty} \frac{2^n e^{-8}}{n!} (x+4)^n$

$$= e^{-8} + 2e^{-8}(x+4) + 2e^{-8}(x+4)^2 + \frac{4}{3}e^{-8}(x+4)^3 + \cdots$$

<u>Radius of convergence</u>:

ratio test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}\cancel{e^{-8}}(x+4)^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n\cancel{e^{-8}}(x+4)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2(x+4)}{n+1}\right| = 0$$

so $L < 1$ for all x so $R = \infty$

$IOC = (-\infty, \infty)$

ex 3/ e^{2x} about $x=0$.

↳ use known series:

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \quad \text{for all } u$$

so $$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n \quad \text{for all } x$$

ex 4/ x^2e^{2x} about $x=0$.

$$x^2e^{2x} = x^2\sum_{n=0}^{\infty}\frac{2^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{2^n}{n!}x^{n+2} \quad \text{for all } x$$

ex 5/ e^{x^2} about $x=0$: $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}(x^2)^n = \sum_{n=0}^{\infty}\frac{1}{n!}x^{2n}$ for all x

$$= 1 + x^2 + \frac{1}{2}x^4 + \cdots$$