Math 143 -

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Differentiation and Integration of Power Series

goal: get power series expansions for more and more functions (and thus, polynomial approximations of higher and higher degree)

Thm. If $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \cdots$

$$= \sum_{n=0}^{\infty} c_n (x-a)^n$$

has a radius of convergence R then $f(x)$ is differentiable (and therefore continuous) on $(a-R, a+R)$ and

① $f'(x) = 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$
with radius of convergence R

② $\int f(x)\,dx = C + c_0(x-a) + \frac{1}{2}c_1(x-a)^2 + \frac{1}{3}c_2(x-a)^3 + \cdots$

constant of integration

$$= C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

with radius of convergence R

<u>notes</u>: a) so integration / differentiation of power series is term-by-term

b) the radius of convergence is preserved

c) the interval of convergence might change, so still need to check endpoints

ex/ Find a power series expansion for $\frac{1}{(1-x)^2}$ about $x=0$.

<u>note</u> ① $\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots}$

$a=1$
$r=x$
$|x|<1$

② $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$

<u>check</u>: $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1-x)^{-1}$

$= -1\cdot(1-x)^{-2}\cdot(-1)$

chain rule

$= (1-x)^{-2} = \frac{1}{(1-x)^2}$

Putting these together, we get

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) \underset{\underset{R=1}{\uparrow}}{=} \frac{d}{dx}\left(1+x+x^2+x^3+\cdots\right)$$

$$= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \cdots$$

$$\underset{\underset{R=1}{\uparrow}}{=} 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$$

So the power series expansion for $g(x) = \frac{1}{(1-x)^2}$ at $x=0$

is $\sum_{n=0}^{\infty} (n+1) x^n$ and the radius of conv is $R=1$

and $IOC = (-1, 1)$

(since endpts $x = \pm 1$ DIV)

ex 2/ Find a power series expansion for $\ln(1+x)$ about $x=0$.

note: ① $\frac{1}{1+x} = \frac{1}{1-(-x)} \underset{\underset{|x|<1}{\uparrow}}{=} 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$

$\boxed{R=1}$

$IOC = (-1,1)$
(geom. ⇒ don't check endpts)

② $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$

so $\ln(1+x) = \int \frac{1}{1+x}\,dx$ up to a constant

R=1

$$\int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx$$

$$= \int 1\,dx - \int x\,dx + \int x^2\,dx - \int x^3\,dx + \cdots$$

$$= C + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \qquad R=1$$

constant of integration

① $\ln(1+x) = C + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

plug in $x=0$: LHS $= \ln(1+0) = \ln(1) = 0$

RHS $= C + 0 - \frac{1}{2}0^2 + \frac{1}{3}0^3 - \frac{1}{4}0^4 + \cdots = C$

so $C=0$

so $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \qquad R=1$$

② endpoints :

$\underline{x=1}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 1^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ CONV by AST

(alt. harmonic series)

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots = \ln(1+1) = \ln(2) \simeq .69314718\ldots$

$\underline{x=-1}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$

$-1\cdot$ harmonic series

$\Rightarrow$ DIV

$$x^a x^b = x^{a+b}$$

$$(-1)^{n-1}(-1)^n = (-1)^{n-1+n}$$

n	1	2	3	4	5	
$2n-1$	1	3	5	7	9	odd numbers!
$(-1)^{2n-1}$	-1	-1	-1	-1	-1	

$$\boxed{\begin{array}{l} \ln(1+x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \\ R=1 \qquad IOC = (-1, 1] \end{array}}$$

ex3/ Find a power series representation of $\arctan(x)$ about $x=0$.

$$\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1\cdot x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x^2)^n$$

note: ① $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$

$R=1$

② $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$

$\arctan(x) = \int \frac{1}{1+x^2} dx$ up to a constant

$= \int (1 - x^2 + x^4 - x^6 + x^8 - \cdots) \, dx$

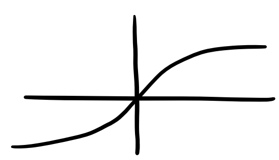
GST $\Rightarrow$ $R=1$

$= C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$

THM $\Rightarrow$ $R=1$

$= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

	n	0	1	2	3	4	5	6	7	
even	$2n$	0	2	4	6	8	10	12	14	...
odd	$2n+1$	1	3	5	7	9	11	13	15	...
	$(-1)^n$	+1	-1	+1	-1	...				



$y = \arctan(x)$

① To find C, plug $x=0$

LHS $= \arctan(0) = 0$

RHS $= C + 0 - \frac{1}{3}0^3 + \frac{1}{5}0^5 - \frac{1}{7}0^7 + \cdots = C$

so $C=0$

so $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ $\quad R=1$

② endpoints:

$x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(-1)^{2n+1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ CONV by AST

$(-1)^{odd} = -1$

$x=1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(1)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ CONV by AST

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$\boxed{\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots}$$

$R=1$ $\quad$ IOC $[-1, 1]$